# 7. UNSTEADY TRANSONIC POTENTIAL FLOW

**Preliminary Remarks**: In the previous chapters we looked at the steady transonic potential flow problem. In this chapter we study the unsteady transonic potential flow problem.

(a) Unsteady transonic potential flows are governed by nonlinear physical phenomena, and differ in this respect from their counterparts in incompressible unsteady aerodynamics. This nonlinearity usually means that it is no longer possible to decompose the total unsteady aerodynamic forces into components using Fourier Series or Transform techniques and analyze the effect of each component on the structural response of the aircraft wing and control surfaces. Many of the classical aeroelastic methods for analyzing aeroelastic phenomena are no longer applicable in unsteady transonic flow.

(b) Unsteady transonic potential flows are also characterized by the long lag between the motion of and aerodynamic surface, and the response of the airflow. This long time lag can lead to deterioration in the flutter speed as the freestream Mach number increases.

(c) Single-degree of freedom flutter can occur in transonic flow as a result of this lag. In incompressible and subsonic flows, flutter usually occurs as a result of the interaction between two modes (e.g. pitching and plunging).

Because of these reasons, unsteady transonic potential flow is one of the least understood and most interesting problems facing researchers today.

In this chapter we will derive the equations of motion for the unsteady potential flow, derive the transonic small disturbance approximation, and also derive the full potential equation in a moving coordinate system. The boundary conditions at the airfoil surface, at the wake, and on the far field boundaries will be derived. Only a semi-implicit time-marching solution procedure for the TSD equation (unsteady) will be discussed in detail.

## GOVERNING EQUATIONS

We start with the continuity equation

$$\rho_t + (\rho u)_x + (\rho v)_y = 0 \qquad (7.1)$$

Upon introduction of a velocity potential $\phi$ (or perturbation potential $\varphi$)

$$u = \phi_x = \varphi_x + 1$$

$$v = \phi_y$$

and the continuity equation becomes

$$\rho_t + (\rho\phi_x)_x + (\rho\phi_y)_y = 0$$

The density is related to the velocity (and hence the potential $\phi$) through the energy equation. Our next task is, therefore, to derive the energy equation for unsteady flows. We start with the u-momentum and v-momentum equation

$$(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0$$

or

$$\rho(\phi_t)_x + \rho u u_x + u[(\rho u)_x + (\rho v)_y + \rho_t] + \rho u v_y + p_x = 0$$

The terms within the square brackets are zero because of the continuity equation. Thus the u-momentum equation becomes

$$(\phi_t)_x + \left(\frac{u^2}{2} + \frac{v^2}{2}\right)_x + \frac{1}{\rho}\frac{dp}{dx} = 0$$

In deriving the above equation we made use of the irrotationality relationship

$$u_y = v_x.$$

Also, for irrotational, inviscid flows,

$$p = C\rho^\gamma \qquad \text{or} \qquad \rho = \rho_\infty \left(\frac{p}{p_\infty}\right)^{\frac{1}{\gamma}}$$

thus

$$\frac{1}{\rho} p_x = \frac{\gamma}{\rho_\infty} \frac{(p_\infty)^{\frac{1}{\gamma}}}{\gamma - 1} \frac{d}{dx}\left\{ p^{\left(-\frac{1}{\gamma}+1\right)} \right\}$$

Thus, the u-momentum equation becomes

$$\frac{\partial}{\partial x}\left[\phi_t + \frac{1}{2}(u^2 + v^2) + \frac{\gamma}{\gamma - 1}\frac{(p_\infty)^{\frac{1}{\gamma}}}{\rho_\infty} p^{\left(1-\frac{1}{\gamma}\right)}\right] = 0$$

or

$$\phi_t + \frac{u^2}{2} + \frac{v^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = f(y) \tag{7.2}$$

where f(y) is an arbitrary function of y or a constant in x.

Doing similar manipulations with the v-momentum equation we get

$$\phi_t + \frac{u^2}{2} + \frac{v^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = F(x) \qquad (7.3)$$

where F(x) is an arbitrary function x. Comparing (7.2) and (7.3), we conclude that

$$f(y) = F(x) = C_1$$

where $C_1$ is an arbitrary constant. Thus the energy equation becomes

$$\phi_t + \frac{u^2}{2} + \frac{v^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = C_1 \qquad (7.4)$$

At the freestream, $\phi_t$ is zero. Thus equation (7.4) becomes

$$\frac{u_\infty^2}{2} + \frac{v_\infty^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_\infty}{\rho_\infty} = C_1 \qquad (7.5)$$

We can eliminate $C_1$ from (7.4) and (7.5). Also noting that

$$u_\infty^2 + v_\infty^2 = q_\infty^2 = 1$$

(by our normalization) and that

$$\gamma \frac{p_\infty}{\rho_\infty} = \gamma R T_\infty = a_\infty^2 = \frac{1}{M_\infty^2}$$

we get

$$\phi_t + \frac{u^2}{2} + \frac{v^2}{2} + \frac{1}{M_\infty^2(\gamma - 1)}\rho^{\gamma - 1} = \frac{1}{2} + \frac{1}{M_\infty^2(\gamma - 1)}$$

or

$$\rho = [1 + \frac{\gamma - 1}{2} M_\infty^2 \{1 - 2\phi_t - \phi_x^2 - \phi_y^2\}]^{\frac{1}{\gamma - 1}} \qquad (7.6)$$

Equation (7.6) is the desired energy equation. Equation (7.1) and (7.6) give two equations and two unknowns ($\phi$ and $\rho$) and provide a complete system of equations for solving the unsteady potential flow problem. The density may be eliminated from the continuity equation as follows. The continuity equation (7.1) becomes upon differentiation by parts,

$$\rho_t + \rho u_x + u\rho_x + \rho v_y + v\rho_y = 0$$

From the energy equation,

$$\rho_t = \frac{1}{\gamma-1}[1+\frac{\gamma-1}{2}M_\infty^2\{1-2\phi_t-\phi_x^2-\phi_y^2\}]^{\left(\frac{1}{\gamma-1}\right)-1}\{\frac{\gamma-1}{2}M_\infty^2(-2\phi_{tt}-2\phi_x\phi_{xt}-2\phi_y\phi_{yt})\}$$

$$= \frac{-1}{\gamma-1}\rho^{2-\gamma}(\gamma-1)M_\infty^2(\phi_{tt}+u\phi_{xt}+v\phi_{yt})$$

$$= -\frac{\rho}{a^2}(\phi_{tt}+u\phi_{xt}+v\phi_{yt})$$

Similarly

$$\rho_x = -\frac{\rho}{a^2}(\phi_{xt}+u\phi_{xx}+v\phi_{xy})$$

$$\rho_y = -\frac{\rho}{a^2}(\phi_{yt}+u\phi_{xy}+v\phi_{yy})$$

The continuity equation becomes

$$(\phi_{tt}+2u\phi_{xt}+2v\phi_{yt}) = (a^2-u^2)\phi_{xx}+(a^2-v^2)\phi_{yy}-2uv\phi_{xy} \tag{7.7}$$

Note that equation (7.7) reduces to the familiar quasilinear form of the steady full potential equation if $\phi$ is not a function of t.

We note that equation (7.7) is a nonlinear equation for $\phi$ involving 3 independent variables x, y, and t. The density does not appear anywhere in equation (7.7). Since equation (7.7) is not in divergence form we suspect that any finite difference solution of (7.7) will lead to a non-conservative solution.

The properties of equation (7.7) are best understood by considering a scaled down version of this equation as shown below. To arrive at this equation, let $\phi$ be only a function of x and t.

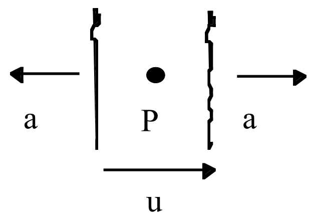
$$(\phi_{tt}+2u\phi_{xt}) = (a^2-u^2)\phi_{xx} \tag{7.8}$$

This equation has the following characteristics

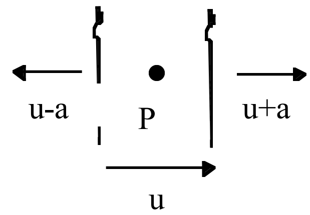
$$\frac{dx}{dt} = \frac{+2u \pm \sqrt{4u^2+4(a^2-u^{2)}}}{2}$$
$$= +u \pm a$$

Thus the characteristics are always real, regardless of whether the flow is subsonic and supersonic. This property is also true of equation (7.7). Equation (7.7) is always hyperbolic, regardless of the local Mach number. Since hyperbolic equations may be solved by a marching procedure (e.g., TSD equation in supersonic regions may be solved by marching one i-line at a time), equation (7.7) may be solved by marching in time, starting with initial conditions ($\phi$ and $\phi_t$) at any starting t-location.

The two characteristics of equation (7.8) have the following physical significance. Consider a point P in the flow field where the local velocity of the flow is u, and the local velocity of sound is a. When a disturbance is produced at the point P, it moves both upstream and downstream with a velocity a relative to the local flow. Relative to a stationary observer, these two waves will have speeds u+a for the downstream traveling wave and u-a for the upstream wave.

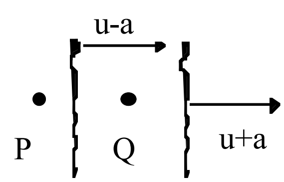



Wave Pattern
Relative to the Flow

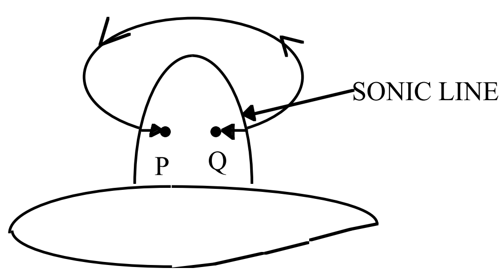



Wave Pattern Relative to
Stationary Observer

The two characteristics of equation (7.8) thus represent the speed of the two waves. The downstream traveling wave is sometimes called the advancing wave, and moves the fastest. The upstream traveling wave is called the receding wave, and moves slowly, particularly in transonic flows (u closest to a). In supersonic flows u>a, and both the waves move downstream, the receding wave moving at the speed u-a. Thus in unsteady supersonic flows, the downstream point Q in the following figure cannot directly influence an upstream point P.




In two-dimensional transonic flows, a downstream point Q can affect an upstream point P only by taking along and winding route such as the one shown in the figure below.




The wave leaves the supersonic region first, travels upstream in the subsonic region, and enters the supersonic region upstream.

The long distances that waves need to travel in transonic flows, plus the fact that the waves are usually receding waves with velocity u-a, explains why unsteady transonic flows have a long time lag between one event (such as the airfoil motion) and another event (such as the development of airloads).

Thus far we have looked at the governing equations (equations (7.1) and (7.6)) and a quasi linear form of the above two equations (equation (7.7)). We next look at the small disturbance approximation to equation (7.7).

**Unsteady Transonic Small Disturbance Equation**

In this section we simplify equation (7.7) using the small disturbance assumptions. From Chapter II, we recall that $u = \phi_x \cong \text{order}(1)$ and $v = \phi_y << 1$ in small disturbance flows.

Equation (7.7) contains the steady part on the right hand side and the unsteady part on the left-hand side. We look at the right hand side first, term by term.

We first note that $\phi_t = \varphi_t$ ; $\phi_{xx} = \varphi_{xx}$ ; $\phi_{yy} = \varphi_{yy}$ . From the energy equation (equation 7.4),

$$\phi_t + \frac{u^2}{2} + \frac{v^2}{2} + \frac{a^2}{\gamma - 1} = \frac{1}{2} + \frac{1}{M_\infty^2(\gamma - 1)}$$

Thus,

$$\begin{aligned} a^2 &= -\varphi_t(\gamma - 1) + \frac{(\gamma - 1)}{2} - \frac{(1 + \varphi_x)^2}{2}(\gamma - 1) - \frac{\varphi_y{}^2}{2}(\gamma - 1) + \frac{1}{M_\infty^2} \\ &\approx -\varphi_t(\gamma - 1) - (\gamma - 1)\varphi_x + \frac{1}{M_\infty^2} \end{aligned}$$

Equation (7.7) requires evaluation of quantities such as $a^2$-$u^2$, $a^2$-$v^2$, etc. We evaluate these quantities here under the small disturbance assumptions.

$$\begin{aligned} a^2 - u^2 &\cong \frac{1}{M_\infty^2} - (\gamma - 1)\varphi_t - (\gamma - 1)\varphi_x - (1 + 2\varphi_x + \varphi_x{}^2) \\ &\cong \frac{1}{M_\infty^2} - (\gamma - 1)\varphi_t + (\gamma + 1)\varphi_x - 1 \end{aligned}$$

Also,

$$\begin{aligned} a^2 - v^2 &\cong \frac{1}{M_\infty^2} - (\gamma - 1)\varphi_t - (\gamma - 1)\varphi_x - \varphi_y{}^2 \\ &\cong \frac{1}{M_\infty^2} \end{aligned}$$

where term 2 is small compared to term 1 in low frequency motions, term 3 is small compared to term 1 and the last term is very close to 0.

Also, $2uv = 2(1+\varphi_x)\varphi_y \cong 0$

Thus, the right hand side of equation (7.7) becomes

$$\frac{1}{M_\infty^2}\left\{1 - M_\infty^2 - (\gamma-1)M_\infty^2\varphi_t - (\gamma+1)M_\infty^2\varphi_x\right\}\varphi_{xx} + \frac{1}{M_\infty^2}\varphi_{yy}$$

to first order accuracy. The term “first order” simply means that we have neglected $\varphi_x^2$ and similar second-degree terms.

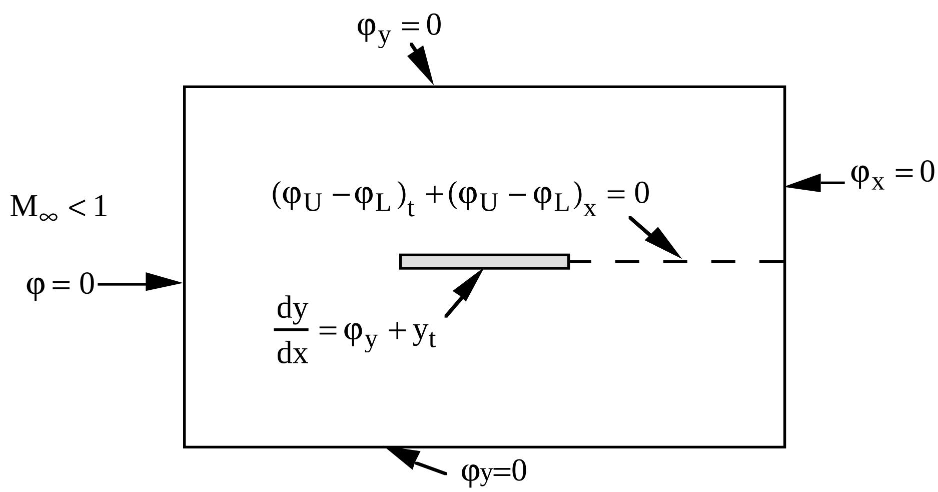
The left-hand side of equation (7.7) is written as

$$\begin{aligned}\phi_{tt} + 2u\phi_{xt} + 2v\phi_{yt} &= \varphi_{tt} + 2(1+\varphi_x)\varphi_{xt} + 2\varphi_y\varphi_{yt} \\ &\cong \varphi_{tt} + 2(1+\varphi_x)\varphi_{xt}\end{aligned}$$

Thus, equation (7.7) becomes subject to small disturbance approximations,

$$M_\infty^2\varphi_{tt} + 2M_\infty^2\varphi_{xt} = \left[(1-M_\infty^2) - (\gamma+1)\varphi_x M_\infty^2 - (\gamma-1)\varphi_t\right]\varphi_{xx} + \varphi_{yy} \qquad (7.9)$$

**Boundary Conditions**: As in the case of the steady small disturbance flow problem, boundary conditions are required at the body surface, at the farfield boundary and at the cut representing the vortex sheet, before equation (7.9) may be solved. We only consider subsonic freestream here.




At the farfield, the disturbances vanish only in steady flow. In unsteady flows, the disturbances generated at the body can travel undamped very long distances, and will eventually reach the farfield boundary. Ideally, these waves should leave the computational boundary. Boundary conditions that allow these waves to pass through, and not be reflected back into the computational domain are called non-reflecting boundary conditions. Currently there is a great deal of research activity in the area of non-reflecting farfield boundary conditions for unsteady potential, Euler and Navier-Stokes equations.

An alternative to the use of non-reflecting boundary conditions is to use the standard boundary conditions ($\varphi = 0$ or $\frac{\partial\varphi}{\partial n} = 0$), and locate the farfield boundaries very far away from the geometry (between 20 and 1000 chord lengths away). If this approach is used, the following farfield boundary conditions are applicable:

$$\varphi = 0 \text{ on the upstream boundary}$$
$$\varphi_y = 0 \text{ on the two lateral boundaries}$$
$$\varphi_x = 0 \text{ on the downstream boundary}$$

We next consider the solid surface. Let P be a point on the solid surface. In unsteady flows, the sold surface is usually moving. Let $(\dot{x}, \dot{y})$ be the velocity components of the solid. Let (u,v) be the velocity components of the fluid at P.

Then the boundary condition at the solid surface is:

slope of the relative velocity vector = slope of the body

$$\frac{v - \dot{y}}{u - \dot{x}} = \frac{dy}{dx}$$

i.e., or

$$\frac{\varphi_y - \dot{y}}{1 + \varphi_x - \dot{x}} = \frac{dy}{dx}$$

If the velocities $(\dot{x}, \dot{y})$ are small, then the above expression may be approximated as $\varphi_y = \frac{dy}{dx} + \dot{y}$ at the solid surface. Thus the velocity of the solid surface enters only through the boundary condition at the solid surface.

As an example, consider the sinusoidal plunging motion of a 10% thick parabolic arc airfoil, with amplitude A. Thus $y = y_{steady}(x) + A\sin\omega t$. Then $\varphi_y = \dot{y} + \frac{dy}{dx} = -.4x + A\omega\cos\omega t$ for the upper surface, and $\varphi_y = \dot{y} + \frac{dy}{dx} = +.4x + A\omega\cos\omega t$ for the lower surface.

We finally consider the boundary conditions at the vortex sheet. In unsteady lifting flows, vorticity is continuously being shed from the trailing edge. Thus, the vortex sheet (or the cut representing the vortex sheet) is an approximation to the vortical wake. It is not just a convenient cut designed to make $\phi$ or $\varphi$non-unique as in the steady flow problems.

At the vortex sheet, the pressures are continuous. This also means that the densities are continuous, or $2\varphi_t + (\varphi_x + 1)^2 + \varphi_y^2$ is continuous. Thus,

$$\left[2\varphi_t + 1 + 2\varphi_x + \varphi_x^2 + \varphi_y^2\right]_{upper} = \left[2\varphi_t + 1 + 2\varphi_x + \varphi_x^2 + \varphi_y^2\right]_{lower}$$

Neglecting second order terms, we get

$$(\varphi_U - \varphi_L)_t + (\varphi_U - \varphi_L)_x = 0$$

or

$$\Gamma_t + \Gamma_x = 0 \text{ where } \Gamma = \varphi_U - \varphi_L$$

The above equation is simply the Vorticity Transport Equation.

The Vorticity Transport Equation tells us that the vorticity in the wake (under small disturbance assumptions) is being convected at the free-stream velocity, regardless of the local Mach number.

Summarizing all of the important equations that we have derived for the unsteady transonic small disturbance flow,

**Governing Equation:**

$$M_\infty^2[\varphi_{tt} + 2\varphi_{xt}] = \left[1 - M_\infty^2 - (\gamma - 1)M_\infty^2\varphi_t - (\gamma + 1)M_\infty^2\varphi_x\right]\varphi_{xx} + \varphi_{yy}$$

**Farfield Boundary Conditions**:

$$\varphi = 0 \text{ upstream}$$
$$\varphi_y = 0 \text{ lateral}$$
$$\varphi_x = 0 \text{ downstream}$$

**Solid Surface**: $\varphi_y = \frac{dy}{dx} + \dot{y}$

**Vortex Sheet**: $\Gamma_t + \Gamma_x = 0 \text{ where } \Gamma = \varphi_U - \varphi_L$

The above equations are simpler than the original set of equations we started with. An additional level of simplification is achieved using the “low frequency” approximation.

**Characteristics of the Low-Frequency Transonic Small Disturbance Equation**

The 1-D analog of the low-frequency transonic small disturbance equation is given by

$$2M_\infty^2\varphi_{xt} = A\varphi_{xx} \text{ where } A = 1 - M_\infty^2 - (\gamma + 1)M_\infty^2\varphi_x$$

Let us freeze the coefficient A; i.e. Assume A is not a function of the reduced potential as far as the following analysis is concerned. Substituting a trial solution $\varphi = f(px + qt)$, we get

$$2M_\infty^2 pqf'' = Ap^2 f''$$

or

$$\frac{q}{p} = \frac{A}{2M_\infty^2} = \frac{1 - M_\infty^2 - (\gamma+1)M_\infty^2\varphi_x}{2M_\infty^2}$$

$\varphi$ is constant along the lines px+qt=constant. These lines are thus the characteristics of the our equation, and the slope of the characteristic is given by $\frac{dx}{dt} = -\frac{p}{q} = -\frac{\left(1 - M_\infty^2 - (\gamma+1)M_\infty^2\varphi_x\right)}{2M_\infty^2}$. The slope represents the speed with which the disturbances travel in "low-frequency small disturbance" flow. We notice that the waves travel upstream (dx/dt negative) whenever the flow is subsonic ($1 - M_\infty^2 - (\gamma+1)M_\infty^2\varphi_x$ >0). The waves travel in the downstream direction at supersonic points.

Thus, the low-frequency approximation retains the receding wave found in the quasilinear formulation. It does not predict and advancing wave however. This is because the advancing wave within the low-frequency small disturbance theory framework has infinite speed and travels downstream instantaneously.

**Discretization of the Low-Frequency Small Disturbance Equation**

Just as it was important to discretize the steady TSD equation in conservation form, it is also important that the low-frequency small disturbance equation be also discretized in conservation form. Non-conservative forms can lead to wrong shock speeds and numerical solutions that are dependent on non-physical phenomena such as grid spacing, time-step size, etc.

Any equation written as

$$(\ )_t + (\ )_x + (\ )_y = 0$$

can be usually discretized in conservation form. As discretized equation is said to be conservative if, upon multiplication by $\Delta t \Delta x \Delta y$ and summation over all the nodes and all the time steps, the summation goes to zero.

$$\sum_{t=0}^{t=n\Delta t} \Delta t \sum_{i,j} (\Delta x_{i+1} + \Delta x_i)(\Delta y_{j+1} + \Delta y_j)(\text{Eqn}) = 0$$

The right hand side of the low-frequency small disturbance equation is familiar from our adventures with the TSD equation. This expression may be written as

$$2\left[\left(1 - \mu_{ij}\right)\frac{P_{i+\frac{1}{2},j} - P_{i-\frac{1}{2},j}}{\Delta x_{i+1} + \Delta x_i} + \mu_{i-1,j}\frac{P_{i-\frac{1}{2},j} - P_{i-\frac{3}{2},j}}{\Delta x_{i+1} + \Delta x_i} + \frac{Q_{i,j+\frac{1}{2}} - Q_{i,j-\frac{1}{2}}}{\Delta y_{j+1} + \Delta y_j}\right]$$

While solving the unsteady low-frequency TSD equation we will be solving this equation one time step at a time. Thus, if we knew the solution $\varphi$ at a time level 'n', our goal is to find the solution at the next time level 'n+1', and then the next time level 'n+2', etc.

The above expression may be therefore written at the know time level n; at the new, unknown time level n+1; or at two-different time levels - the terms involving P at time level 'n' and the terms involving Q at time level 'n+1', for example. We will use this third choice. That is, the right hand side of the unsteady low-frequency TSD equation is written as

$$2\left[\left(1-\mu_{ij}\right)\frac{P^n_{i+\frac{1}{2},j}-P^n_{i-\frac{1}{2},j}}{\Delta x_{i+1}+\Delta x_i}+\mu_{i-1,j}\frac{P^n_{i-\frac{1}{2},j}-P^n_{i-\frac{3}{2},j}}{\Delta x_{i+1}+\Delta x_i}+\frac{Q^{n+1}_{i,j+\frac{1}{2}}-Q^{n+1}_{i,j-\frac{1}{2}}}{\Delta y_{j+1}+\Delta y_j}\right]$$

where $P=\left[1-M_\infty^2-\frac{(\gamma+1)}{2}M_\infty^2\varphi_x\right]\varphi_x$ and $Q=\varphi_y$.

It can be demonstrated that upon multiplication by $\Delta t\left(\Delta x_{i+1}+\Delta x_i\right)\left(\Delta y_{j+1}+\Delta y_j\right)$, and summation over all i and j, and over all time steps n=0,N, the right hand side of the discretized low-frequency TSD (independent of the left hand side), goes to zero. The right hand side is in conservation form. We next consider $2M_\infty^2\varphi_{xt}$ on the left hand side. Consider $\varphi^{n+1}_{i-1,j}$ or $\varphi(x_{i-1},y_j,(n+1)\Delta t)$. This quantity may be written as a Taylor series about $\varphi(x_i,y_j,n\Delta t)$ as

$$\varphi^{n+1}_{i-1,j}=\varphi^n_{i,j}-\Delta x_i\left(\varphi_x\right)^n_{i,j}+\Delta t\left(\varphi_t\right)^n_{i,j}-\Delta x_i\Delta t\left(\varphi_{xt}\right)^n_{i,j}+\text{Higher Order Terms}$$

or

$$\varphi^{n+1}_{i-1,j}=\varphi^n_{i,j}-\left(\varphi^n_{i,j}-\varphi^n_{i-1,j}\right)+\left(\varphi^{n+1}_{i,j}-\varphi^n_{i,j}\right)-\Delta x_i\Delta t\left(\varphi_{xt}\right)_{ij}+\text{Higher Order Terms}$$

Thus, $\varphi_{xt}\big|^n_{i,j}=\dfrac{\varphi^{n+1}_{i,j}-\varphi^{n+1}_{i-1,j}+\varphi^n_{i-1,j}-\varphi^n_{i,j}}{\Delta x_i\Delta t}$.

The discretized low-frequency TSD equation becomes

$$\frac{2M_\infty^2(\varphi^{n+1}_{i,j}-\varphi^{n+1}_{i-1,j}+\varphi^n_{i-1,j}-\varphi^n_{i,j})}{\Delta x_i\Delta t}=\text{RHS}$$

Unfortunately, if we multiply the above equation by $\Delta t(\Delta x_{i+1}+\Delta x_i)(\Delta y_{j+1}+\Delta y_j)$ and sum over all nodes and all time levels, the right hand side goes to zero as noted earlier, but the left-hand side does not. A simple, effective fix to this problem is to replace $\Delta x_i$ on the denominator by $\frac{\Delta x_{i+1}+\Delta x_i}{2}$. Thus the final equation becomes

$$\frac{4M_\infty^2(\varphi_{i,j}^{n+1}-\varphi_{i-1,j}^{n+1}+\varphi_{i-1,j}^{n}-\varphi_{i,j}^{n})}{(\Delta x_i+\Delta x_{i+1})\Delta t}=$$

$$2\left[(1-\mu_{ij})\frac{P_{i+\frac{1}{2},j}^{n}-P_{i-\frac{1}{2},j}^{n}}{\Delta x_{i+1}+\Delta x_i}+\mu_{i-1,j}\frac{P_{i-\frac{1}{2},j}^{n}-P_{i-\frac{3}{2},j}^{n}}{\Delta x_{i+1}+\Delta x_i}+\frac{Q_{i,j+\frac{1}{2}}^{n+1}-Q_{i,j-\frac{1}{2}}^{n+1}}{\Delta y_{j+1}+\Delta y_j}\right] \quad (7.10)$$

The above form of discretization was proposed and used by Ballhaus and Lomax (First International Conference on Numerical Methods in Fluid Dynamics Proceedings, 1972). Equation (7.10) may be solved one line at a time as done in the SLOR scheme. First of all, all the quantities at the known level are taken to the right hand side, and the quantities at time level ‘n+1’ are brought to the left-hand side. Thus at every point (i,j) we get

$$\frac{4M_\infty^2(\varphi_{i,j}^{n+1}-\varphi_{i-1,j}^{n+1})}{(\Delta x_i+\Delta x_{i+1})\Delta t}-2\left[+\frac{Q_{i,j+\frac{1}{2}}^{n+1}-Q_{i,j-\frac{1}{2}}^{n+1}}{\Delta y_{j+1}+\Delta y_j}\right]$$

$$=2(1-\mu_{ij})\frac{P_{i+\frac{1}{2},j}^{n}-P_{i-\frac{1}{2},j}^{n}}{\Delta x_{i+1}+\Delta x_i}+2\mu_{i-1,j}\frac{P_{i-\frac{1}{2},j}^{n}-P_{i-\frac{3}{2},j}^{n}}{\Delta x_{i+1}+\Delta x_i}+\frac{4M_\infty^2(\varphi_{i,j}^{n}-\varphi_{i-1,j}^{n})}{(\Delta x_i+\Delta x_{i+1})\Delta t}$$

The equation is written at every point i,j keeping i fixed. At points adjacent to the solid, $\varphi_y$ on the left hand side is replaced by $\left(\frac{dy}{dx}\right)_{mean}+\dot{y}$. A tridiagonal system of equations for $\phi_{i,j}^{n+1}$ results at each i, for j=2,3,.... This system may be solved by Thomas Algorithm. At the wake care must be taken while evaluating $\varphi_y$, as discussed previously.

Let $R_{i,j}^{n}$ denote the right hand side of the discretized low-frequency TSD equation. Then we can write this equation as

$$t_{ij}C_{ij}^{n+1}+v_{ij}C_{ij-1}^{n+1}+w_{ij}C_{ij+1}^{n+1}=R_{ij}^{n}$$

$$t_{ij} = \frac{4M_\infty^2}{(\Delta x_{i+1} + \Delta x_i)\Delta t} + \frac{2}{(\Delta y_{j+1} + \Delta y_j)}\left[\frac{1}{\Delta y_j} + \frac{1}{\Delta y_j + 1}\right]$$

where
$$v_{ij} = -\frac{2}{(\Delta y_{j+1} + \Delta y_j)}\left[\frac{1}{\Delta y_j}\right]$$

$$w_{ij} = -\frac{2}{(\Delta y_{j+1} + \Delta y_j)}\left[\frac{1}{\Delta y_{j+1}}\right]$$

The above equation written one equation per point leads to a system of equations solved easily by the Thomas Algorithm.

The above equation is called a semi-implicit equation because only part of the spatial derivatives (the $Q_y$ terms) are at the new time level, and the rest of the derivatives are at the old time level. As a rule, fully-implicit scheme (where both the $P_x$ and $Q_y$ terms are at the level 'n+1') are "unconditionally" stable, i.e., one can take very large time steps and march rapidly in time. Semi-implicit schemes are conditionally stable.

By using a von Neumann stability analysis, one can show that the semi-implicit procedure just described is stable for time steps $\Delta t$ obeying the criterion:

$$\Delta t \le \frac{M_\infty^2(\Delta x_{i+1} + \Delta x_i)}{\left|1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \varphi_x\right|}$$

Thus arbitrarily large $\Delta t$ values can not be used. Despite this limitation, the semi-implicit algorithm just described is a useful algorithm. The computer program developed in Chapter II for the steady TSD equation can be easily modified to solve the unsteady low-frequency TSD equation.

For a discussion on fully implicit schemes for the low-frequency TSD equation see Ballhaus and Steger: NASA TMX-73,082, November 1975.