

LOW-SPEED AERODYNAMICS EQUATIONS OVERVIEW

Continuity Equation $\frac{\partial}{\partial t} \iiint_{CV} \rho dV + \iint_{CS} \rho (\vec{V} \cdot \vec{n}) dA = 0$

Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r V_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho V_\theta)}{\partial \theta} + \frac{\partial(\rho V_z)}{\partial z} = 0$$

Momentum Equation

$$\iiint_{CV} \rho \vec{f} dV + \vec{F}_{visc} - \iint_{CS} P \vec{n} dA + \sum \vec{R} = \frac{\partial}{\partial t} \iiint_{CV} \rho \vec{V} dV + \iint_{CS} \rho \vec{V} (\vec{V} \cdot \vec{n}) dA$$

Cartesian coordinates:

$$\begin{aligned} x\text{-direction} \quad & \frac{\partial(\rho u)}{\partial t} + \vec{\nabla} \cdot (\rho u \vec{V}) + \frac{\partial P}{\partial x} - \rho f_x - (\tilde{F}_{visc})_x = 0 \\ y\text{-direction} \quad & \frac{\partial(\rho v)}{\partial t} + \vec{\nabla} \cdot (\rho v \vec{V}) + \frac{\partial P}{\partial y} - \rho f_y - (\tilde{F}_{visc})_y = 0 \\ z\text{-direction} \quad & \frac{\partial(\rho w)}{\partial t} + \vec{\nabla} \cdot (\rho w \vec{V}) + \frac{\partial P}{\partial z} - \rho f_z - (\tilde{F}_{visc})_z = 0 \end{aligned}$$

Substantial Derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

Streamlines

$$d\vec{s} \times \vec{V} = 0$$

Cartesian coordinates:

$$w dy - v dz = 0$$

$$u dz - w dx = 0$$

$$v dx - u dy = 0$$

for 2-D flow:

$$\frac{dy}{dx} = \frac{v}{u}$$

Vorticity $\vec{\xi} \equiv 2\vec{\omega} = \vec{\nabla} \times \vec{V} = \text{curl } \vec{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}$

polar coordinates:

$$2\vec{\omega}_z = \left[\frac{\partial(V_\theta)}{\partial r} + \frac{1}{r} \left(V_\theta - \frac{\partial V_r}{\partial \theta} \right) \right] \hat{k}$$

Bernoulli Equation

$$\text{no body forces : } P + \frac{1}{2} \rho V^2 = P_0 = \text{constant}$$

$$\text{incl. body forces : } P + \frac{1}{2} \rho V^2 + \rho g z = \text{constant}$$

$$\text{Pressure Coefficient } C_p \equiv \frac{(P - P_\infty)}{\frac{1}{2} \rho_\infty V_\infty^2} = 1 - \left(\frac{V}{V_\infty} \right)^2 \text{ for incompressible flow}$$

Circulation

$$\Gamma \equiv -\oint_C \vec{V} \cdot d\vec{s} = -\iint_S (\vec{\nabla} \times \vec{V}) \cdot \vec{n} dA$$

Note : Line integral in counter - clockwise direction

Velocity Potential and Stream Function

Cartesian coordinates:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$
$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

polar coordinates:

$$V_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$
$$V_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

$$\text{Laplace's Equation: } \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\text{Lifting Flow Over a Cylinder } \psi = (V_\infty r \sin \theta) \left(1 - \frac{R^2}{r^2} \right) + \frac{\Gamma}{2\pi} \ln \frac{r}{R}$$

$$\text{with cylinder radius } R = \sqrt{\frac{\kappa}{2\pi V_\infty}}$$

$$\text{Kutta-Joukowski Theorem } L' = \rho_\infty V_\infty \Gamma$$

Elementary Flows

$$\text{(Note: } d[\tan^{-1} z] = \frac{1}{1+z^2} dz \text{)}$$

Uniform Flow:

$$\begin{aligned} \phi &= V_{\infty} r \cos \theta = V_{\infty} x & \psi &= V_{\infty} r \sin \theta = V_{\infty} y \\ u &= V_{\infty} & v &= 0 \end{aligned}$$

Source/Sink Flow:

$$\begin{aligned} \phi &= \frac{\Lambda}{2\pi} \ln r = \frac{\Lambda}{2\pi} \ln \sqrt{x^2 + y^2} & \psi &= \frac{\Lambda}{2\pi} \theta = \frac{\Lambda}{2\pi} \tan^{-1} \left(\frac{y}{x} \right) \\ V_r &= \frac{\Lambda}{2\pi r} & V_{\theta} &= 0 \end{aligned}$$

Vortex Flow:

$$\begin{aligned} \phi &= -\frac{\Gamma}{2\pi} \theta = -\frac{\Gamma}{2\pi} \tan^{-1} \left(\frac{y}{x} \right) & \psi &= \frac{\Gamma}{2\pi} \ln r = \frac{\Gamma}{2\pi} \ln \sqrt{x^2 + y^2} \\ V_r &= 0 & V_{\theta} &= -\frac{\Gamma}{2\pi r} \end{aligned}$$

Doublet Flow:

$$\begin{aligned} \phi &= \frac{\kappa \cos \theta}{2\pi r} = \frac{\kappa x}{2\pi x^2 + y^2} & \psi &= -\frac{\kappa \sin \theta}{2\pi r} = -\frac{\kappa y}{2\pi x^2 + y^2} \\ V_r &= -\frac{\kappa \cos \theta}{2\pi r^2} & V_{\theta} &= -\frac{\kappa \sin \theta}{2\pi r^2} \end{aligned}$$

Thin Airfoil Theory

$$c_l = 2\pi \left[\alpha + \frac{1}{\pi} \int_{\theta_0=0}^{\pi} \frac{dz}{dx} (\cos \theta_0 - 1) d\theta_0 \right] \quad x = \frac{c}{2} (1 - \cos \theta_0)$$

$$\alpha_{L_0} = -\frac{1}{\pi} \int_{\theta_0=0}^{\pi} \frac{dz}{dx} (\cos \theta_0 - 1) d\theta_0$$

$$c_{m_{le}} = -\left[\frac{c_l}{4} + \frac{\pi}{4} (A_1 - A_2) \right] \quad c_{m_{ac}} = \frac{\pi}{4} (A_2 - A_1)$$

where

$$A_0 = \alpha - \frac{1}{\pi} \int_{\theta_0=0}^{\pi} \frac{dz}{dx} d\theta_0 \quad A_n = \frac{2}{\pi} \int_{\theta_0=0}^{\pi} \frac{dz}{dx} \cos(n\theta_0) d\theta_0 \text{ for } n \geq 1$$

$$x_{cp} = \frac{c}{4} \left[1 + \frac{\pi}{c_l} (A_1 - A_2) \right] \quad x_{ac} = \frac{c}{4}$$

$$\gamma(\theta) = 2V_{\infty} \left[A_0 \frac{1 + \cos \theta}{\sin \theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta) \right] \quad \xi = \frac{c}{2} (1 - \cos \theta)$$

Trigonometric Identities

$$\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)$$

$$\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta)$$

$$d[\tan^{-1} z] = \frac{1}{1+z^2} dz$$

$$\int \sin^n ax \, dx = \frac{-\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int \sin^{n-2} ax \, dx$$

$$\int \cos^n ax \, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax \, dx$$