Counter-example to a recent result on the stability of nonlinear systems

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Abstract

S.P. Banks and K.J. Mhana (1992) claim that a certain condition is sufficient to ensure global asymptotic stability for a broad class of nonlinear systems. We demonstrate, via a counter-example, that satisfaction of this condition does not imply global asymptotic stability.

Counter-example

A recent paper by S.P. Banks and K.J. Mhana (1992) claims that, if a certain condition holds, one can guarantee global asymptotic stability for a nonlinear system of the form

$$\dot{x} = A(x)x \tag{1}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$, and A is a continuously differentiable matrix-valued function. Specifically, the following claim is made in a remark in Banks and Mhana (1992, p. 187). If, for each x, all the eigenvalues of the matrix $A(x)$ are in the open left half complex plane then, zero is an asymptotically stable equilibrium solution of (1) for all initial states $x(0)$. We now show via a counter-example that this statement is, in general, *false*.

Consider the following two-dimensional nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2 \end{aligned} \tag{2}$$

Let $A(x)$ be given by

$$A(x) = \begin{bmatrix} -1 & x_1^2 \\ 0 & -1 \end{bmatrix}. \tag{3}$$

Then, with this $A(x)$, (2) can be written as (1). Note that $A(\cdot)$ is real analytic and, for each x, all the eigenvalues of $A(x)$ equal -1.

With initial state,

$$\begin{aligned} x_1(0) &= 2 \\ x_2(0) &= 2 \end{aligned} \tag{4}$$

a routine calculation shows that for $t \in [0, T_e)$, $T_e = \ln\sqrt{2}$, the functions

$$x_1(t) = \frac{2x_2(t)}{x_2^2(t) - 2} \qquad \text{and} \qquad x_2(t) = 2e^{-t}, \tag{5}$$

are a solution to (2). As can be easily checked the solution (5) has a finite escape time at $T_e = \ln\sqrt{2}$; this is because $x_1(t)$ grows without bound as $t \to T_e$. Hence, the system (2) is not asymptotically stable for all initial states $x(0)$.

Moreover, the solution (5) is not even in $L_2(0, T_e)$, i.e., it is not square-integrable. A straightforward calculation shows that for $T \in [0, T_e)$,

$$J(T) := \int_0^T (x_1^2(t) + x_2^2(t))dt = \frac{1}{2e^{-2T} - 1} - 2e^{-2T} + 1. \tag{6}$$

Clearly, we have $\lim_{T \to T_e} J(T) = \infty$. Therefore, $x(\cdot)$ is not in $L_2(0, T_e)$.

The system (2) also furnishes a counter-example to Lemma 4.1 and Theorem 4.1 of Banks and Mhana (1992). Indeed, $Q = I$, $B(x) = 0$, and the matrix $A(x)$ of (3) satisfy the assumptions of Lemma 4.1 and Theorem 4.1; however, the conclusions of this lemma and theorem are *false* for the initial state $x(0) = (2, 2)$.

References

BANKS, S.P., AND MHANA, K.J. 1992 Optimal control and stabilization for nonlinear systems, *IMA J. of Mathematical Control & Information*, **9**, 179-196.