Experimental Errors and Uncertainty: An Introduction

Prepared for students in AE 3051

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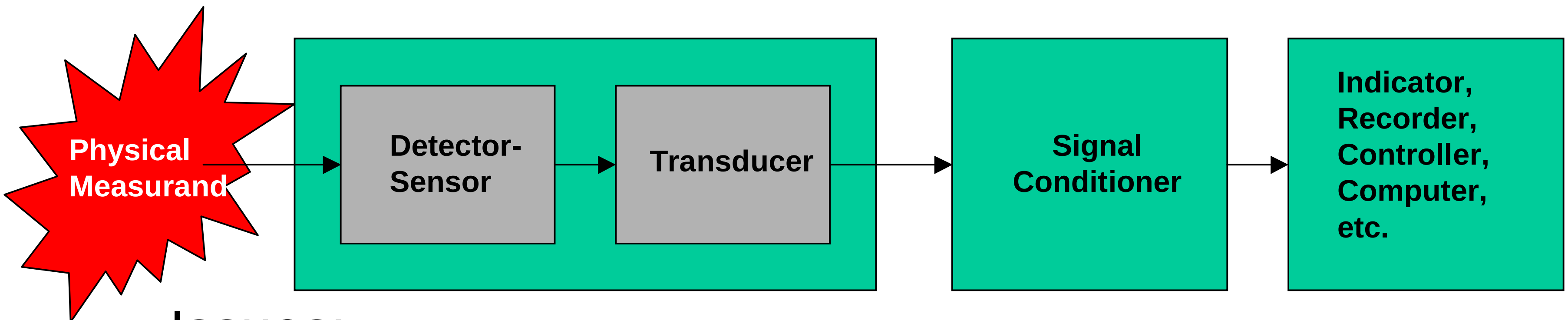
adapted from material made available by J. Craig

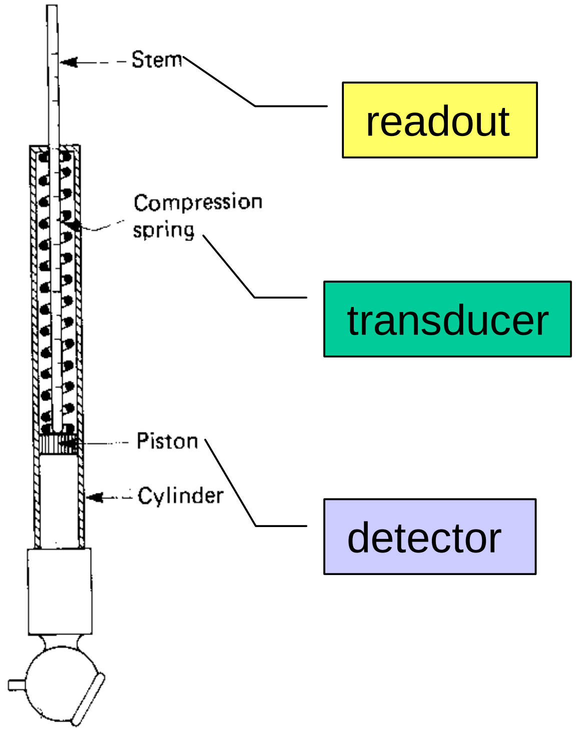
The Measurement System

- Measurements
  - Direct/Indirect comparison (rulers, balance scale, interferometer
  - Calibrated system (odometer, spring scale, pressure gage)
- Measurement System

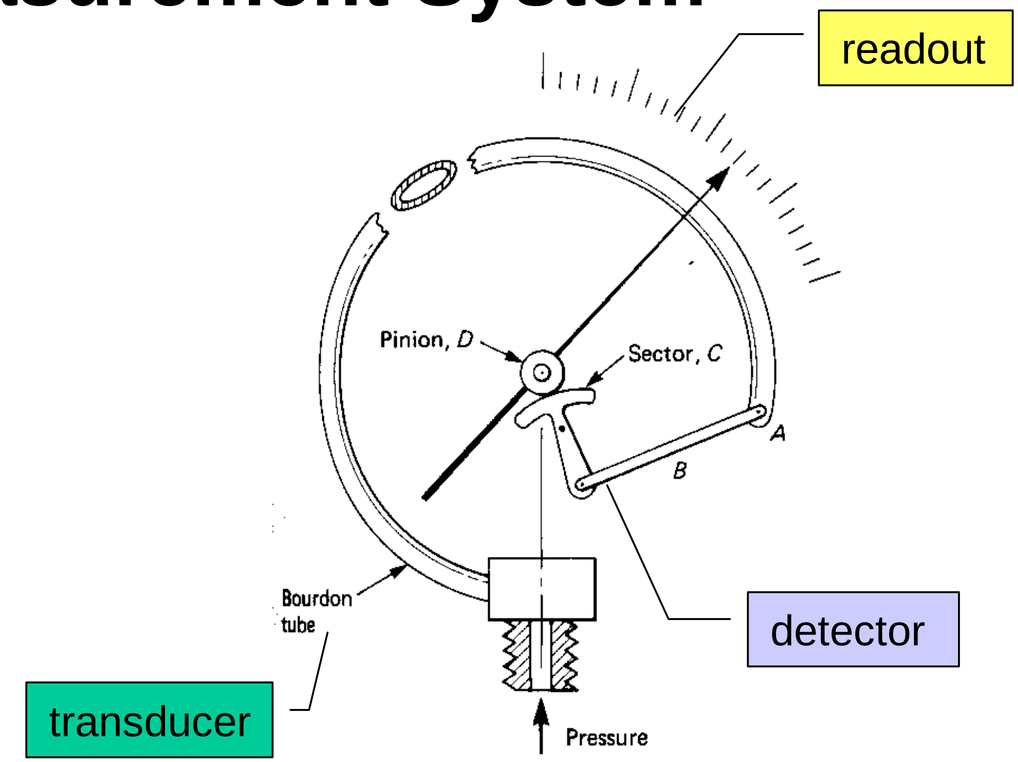


- Issues:
  - Detector/Sensor: device which detects and responds to measurand
  - Transducer: converts measurand to an analog more easily measured (force-displacement-resistance-voltage)
  - Signal Cond.: amplify, filter, integrate, differentiate, convert freq. to voltage, etc.
  - Computer: widely used today

Example of Measurement System



Tire Pressure Gage

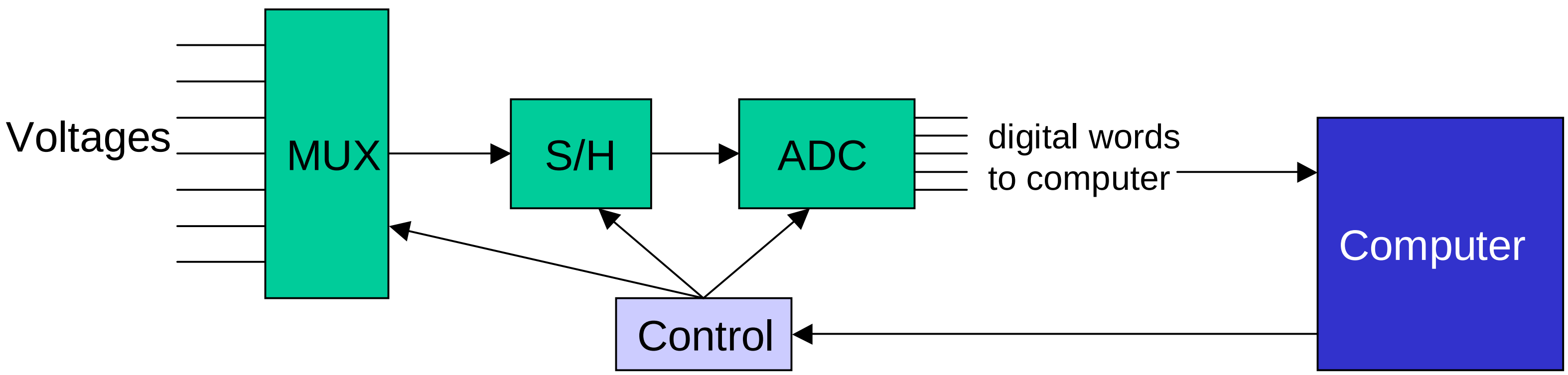


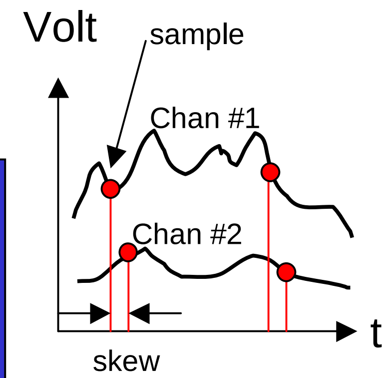
Bourdon Type Pressure Gage

- These are simple mechanical systems
- Issues
  - mechanical vs electrical output
  - analog vs digital
  - calibration
  - accuracy, precision, resolution, sensitivity, linearity, drift, backlash?

Computer Readout Systems

Multichannel Data Acquisition (sequential sampling)





Readings are not recorded simultaneously

- Each channel is read in sequence (10 ms to 1 ms per reading).
- ADC may output a reading that is from 8 to 16 bits in size.
  - *8 bits & bipolar range yields readings from –128 to +127 (2^8=256) corresponding to –Full Scale and + Full Scale*
  - *16 bits & bipolar range yields reading from –32768 to +32767 (2^{16}=65,536) so resolution is about 4½ digits or about 0.01% of Full Scale (not of reading)*
  - *With ±1v range, an 8 bit ADC has a 7.9 mv resoultion (=1/127)*
  - *With ±10v range, a 16 bit ADC has a 0.31 mv resolution (=10/32767)*

Key:
MUX = multiplexer (switch)
S/H = sample & hold (hold voltage while ADC reads)
ADC = analog to digital converter (voltmeter)

Georgia Tech College of Engineering

Experimental Error

- **Error**: all measurements have some uncertainty

$$\text{error} = \varepsilon = x_{meas} - x_{exact}$$

- **Objectives**

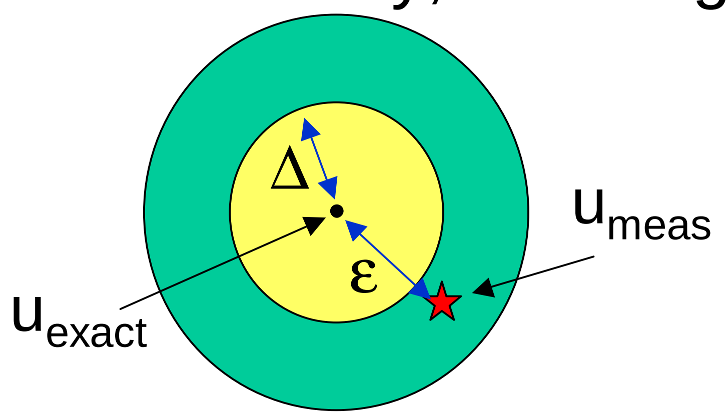
1. **Minimize** error so that uncertainty, u, is:

   $-u \le \varepsilon \le +u$ *at N:1 certainty (a statistical confidence)*

   or $x_{meas} - u \le x_{exact} \le x_{meas} + u$

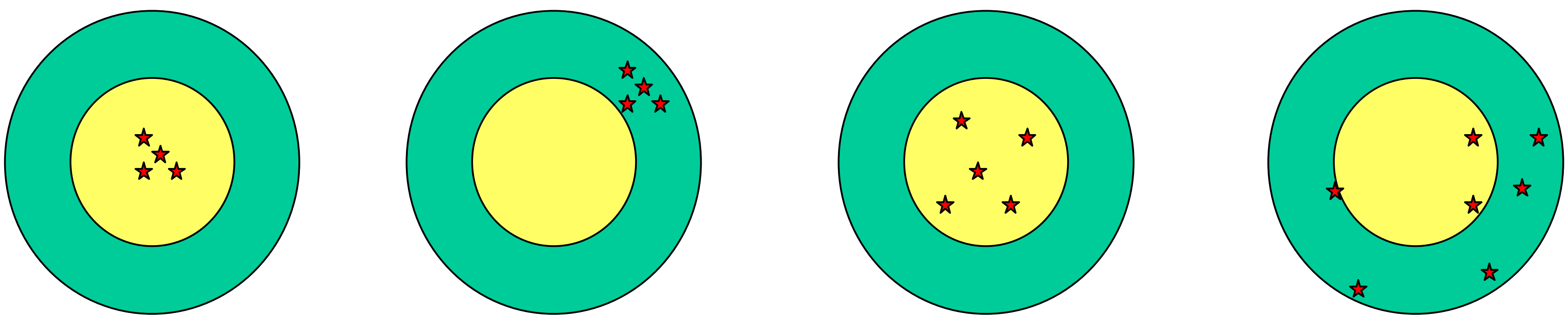
2. **Estimate** error to

   determine reliability, meaningfulness of data



Accuracy and Precision

- **Accuracy:** also called *systematic* or *bias* error
  - denotes something repeatably “wrong” with the measurement or experiment
- **Precision:** also called *random* error or *noise*
  - denotes errors that change randomly each time you try to repeat experiment



Good Accuracy
Good Precision

Good Precision
Poor Accuracy
(can calibrate)

Good Accuracy
Poor Precision
(can average)

Poor Accuracy
Poor Precision

Other Related Terms

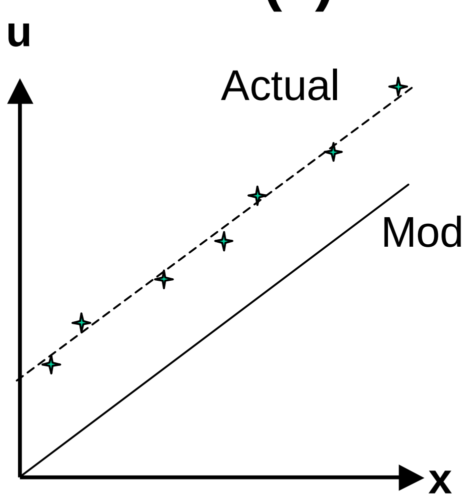
- **Resolution**
  - Smallest increment of change in a system or property that a **measurement device** can reliably capture
- **Sensitivity**
  - Change in a **measurement device**'s output for a unit change in the measured (input) quantity, e.g., volts/Torr for the Baratron
- **Dynamic Range**
  - Maximum output of a **measurement device** divided by its resolution

Accuracy/Systematic Errors

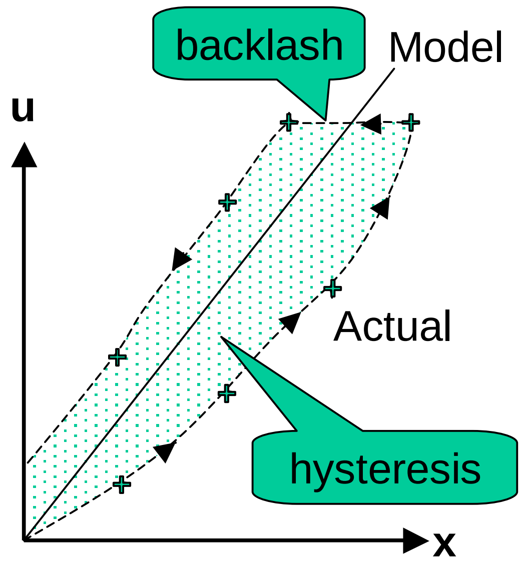
- **Sources**
  - **Measur<u>ing</u> system** errors
    - difference between *model* of measuring system and *reality*
    - could be corrected, e.g., with better model of measurement
  - **Measur<u>ed</u> system** "errors"
    - influence of uncontrolled or unaccounted for variables in the experiment
    - the measured data may be "correct", but may lead to an incorrect model of the object/process being studied

Some Systematic Errors of Measuring Systems

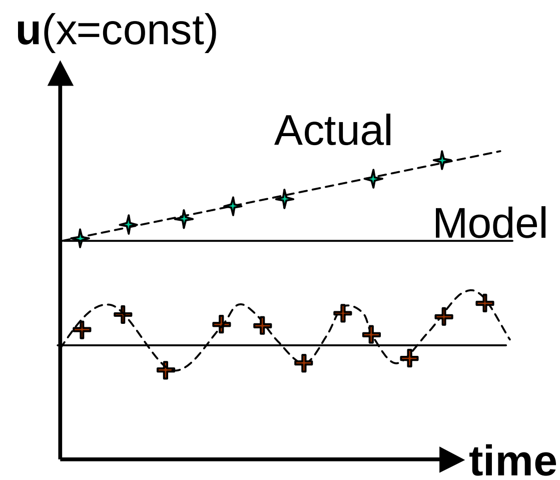
•u=u(x)



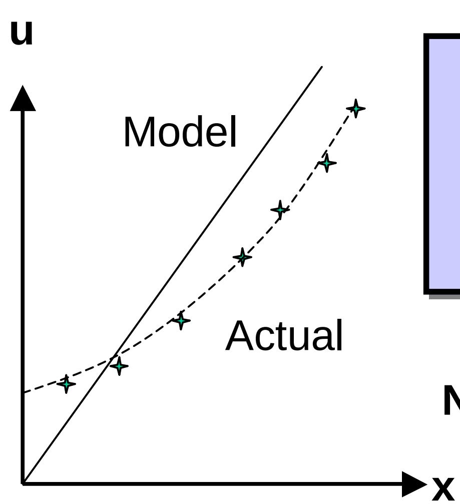
Nonzero offset - Background



Backlash & Hysteresis



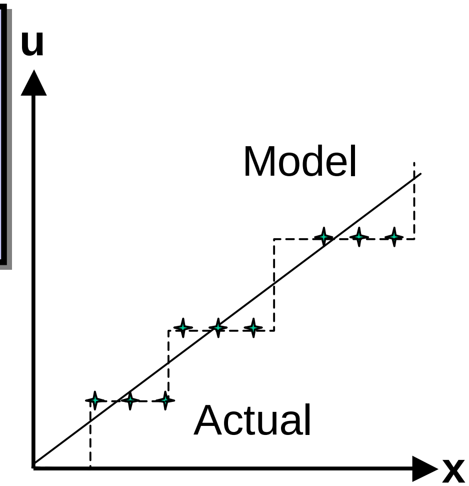
Drift (e.g., offset changing in systematic way with time)



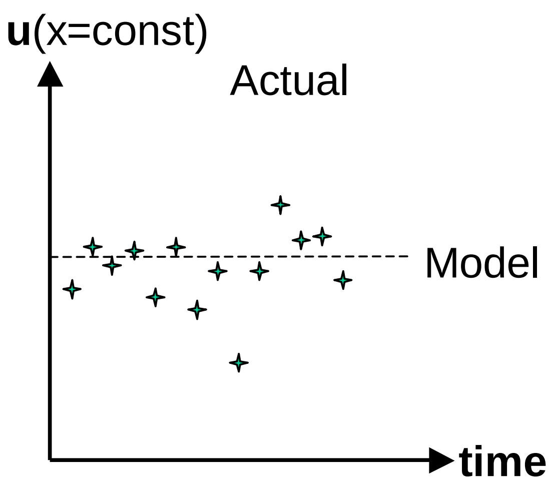
Nonlinearity

Systematic errors can be eliminated/removed if they are known

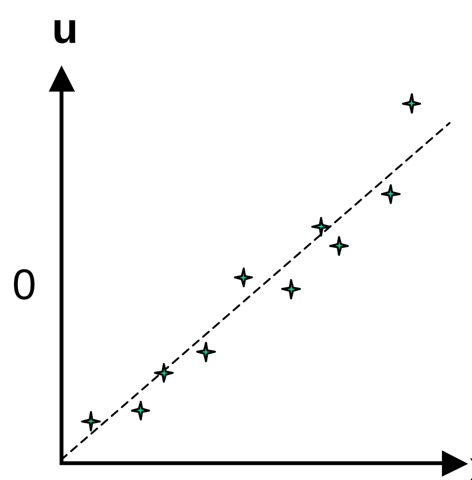
Quantization Error (digitized data – impacts resolution)



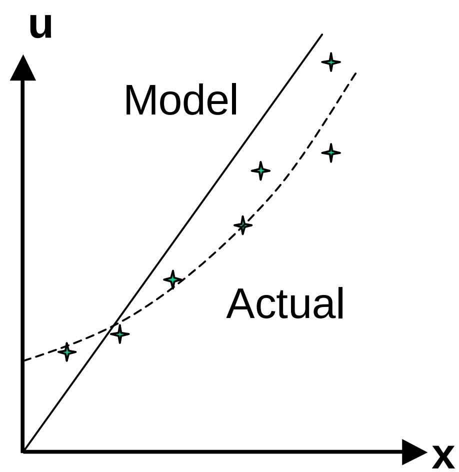
Some Random Errors of Measuring Systems



Background "Noise" (offset changing randomly with time)



Detector "Noise" (random change in sensitivity of device)

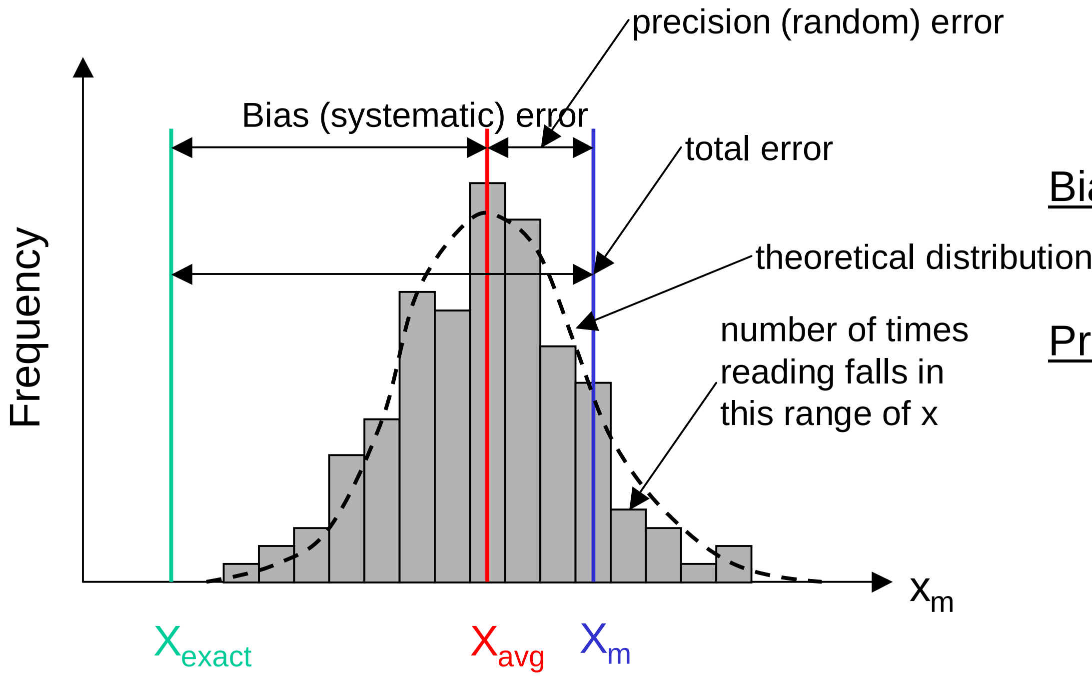


Disturbances – "Noise" (e.g., pickup electrical signals from other sources

After data acquired, nearly impossible to separate random error (noise) sources

Uncertainty: Statistical Approaches

- Probability & statistics provide a way to deal with uncertainty
  - we will cover only a VERY limited introduction



Bias error: systematic errors that could be removed by proper calibration

Precision error: random error - not directly controllable

Bias:
- calibration
- consistent human error
- background

Precision:
- disturbances
- noise
- variable conditions

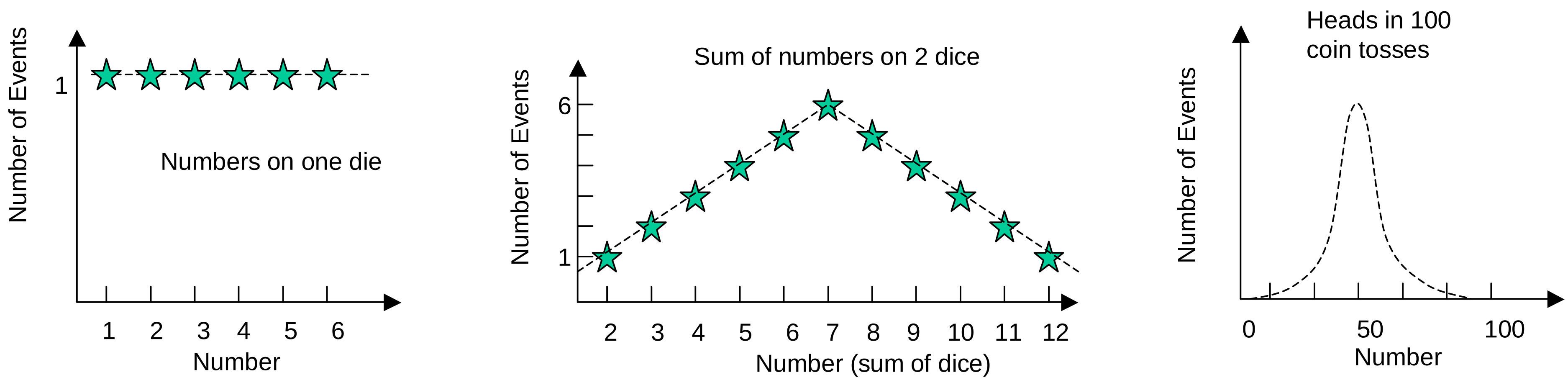
Blunders:
- human error
- software!

Either:
- backlash, hysteresis, friction
- damping
- drift
- variations in test procedure

Basic Concepts in Probability

- **Sample Space**: all possible events or outcomes of an experiment; also can be referred to as a population.
- **Event**: subset of sample space
- **Sample**: finite number taken from population (e.g., 15 turbine blades taken from 8600 produced for XX-300 engine)
  - sample must be randomly selected from population
  - sample may or may not be returned to population before resampling
- **Probability**: likelihood of an event (measured as % of successful events in sample space or population). We can often analytically compute this likelihood based on mathematics applied to the population. $0 \leq P(\text{event}) \leq 1$.
- There are many rules for computing probabilities for different kinds of events; these are the subject of courses and books on probability theory.

Examples



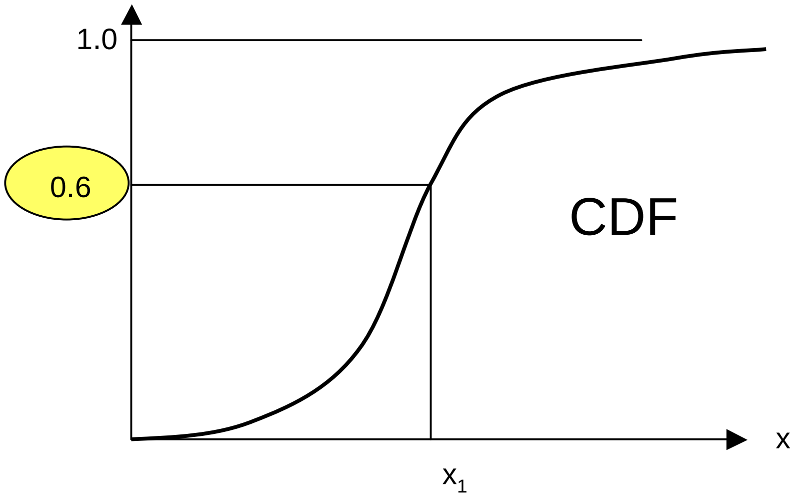
- These happen to be binomial distributions *(repeat experiment N times; each trial is independent of others; each trial is successful or not; probability of success, p, is same for each trial).*

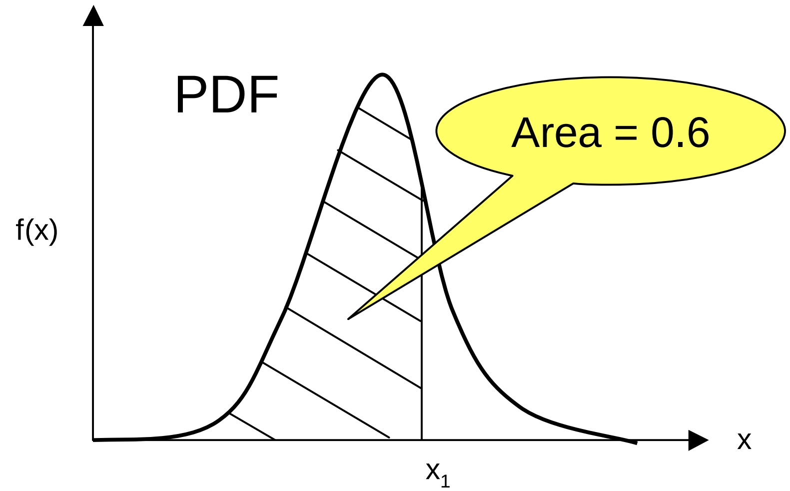
$$P(x \text{ successes in } N \text{ trials}) = \binom{N}{x} p^x (1-p)^{N-x}$$

$$\binom{N}{x} = \frac{N!}{x!(N-x)!}$$

- These are *discrete distributions* but there are also *continuous distributions* that are defined by CDF and PDF curves (next)

Properties of Probability Distribution Functions





CDF = Cumulative Distribution Function = P(x ≤ x_1)

PDF = Probability Distribution Function = f(x)

$$P(x \le x_1) = 0.6$$

also

$$P(x \le \infty) = 1.0$$

$$P(x \le -\infty) = 0$$

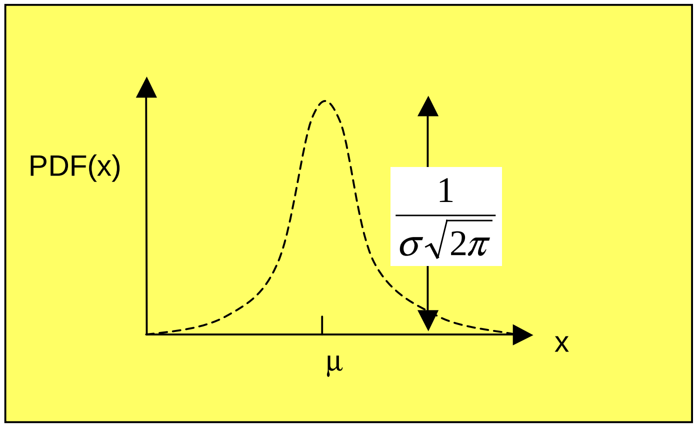
$$f(x) = \frac{d}{dx}(CDF)$$

$$\text{or} \quad CDF = \int f(x)\,dx$$

$$\text{So} \quad P(x \le x_1) = \int_{-\infty}^{x_1} f(x)\,dx$$

$$\text{and} \quad P(x_1 \le x \le x_2) = \int_{x_1}^{x_2} f(x)\,dx$$

Normal/Gaussian Probability Distribution



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

μ = mean
σ^2 = variance
σ = standard deviation

You can use a spreadsheet!

- Uses:
  - When N→∞ for a binomial distribution (e.g., **for very large populations**)
  - For events that are made up from **many independent events** each with any kind of distribution
  - For properties of samples when **# samples is very large** (e.g., sample means)
  - others…

Normal Distributions - Confidence Levels

One Sigma: $\Pr ob \ (\mu - \sigma \le x \le \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(\xi) d\xi = \int_{-1}^{+1} f(z) dz = 0.683$

Two Sigma: $\Pr ob \ (\mu - 2\sigma \le x \le \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} f(\xi) d\xi = \int_{-2}^{+2} f(z) dz = 0.954$

Three Sigma: $\Pr ob \ (\mu - 3\sigma \le x \le \mu + 3\sigma) = 0.997$

Error Level Name	Error Level	Prob. that Error is Smaller	Prob. that Error is Larger
Probable Error	± 0.67 σ	50%	1:2
One Sigma	± σ	68%	~ 1:3
90% error	± 1.65 σ	90%	1:10
"Two" Sigma	± 1.96 σ	95%	1:20
Three Sigma	± 3 σ	99.7%	1:370
Maximum Error	± 3.29 σ	99.9	1:1000
Four Sigma	± 4 σ	99.994%	1:16000
Six Sigma	± 6 σ	99.9999999%	1:1.01e9

Six Sigma is used for many electronic manufacturing processes

Sample Statistics

- What if μ and σ are unknown (as is often the case)?
  - use estimates from measurements, $\overline{x}$ and s

Mean Square

Square of mean

$$Sample \quad mean = \overline{x} = \sum_{i=1}^{N} \frac{x_i}{N}$$

$$Sample \quad variance = s^2 = \sum_{i=1}^{N} \frac{(x_i - \overline{x})^2}{N-1} = \left(\frac{\sum_{i=1}^{N} x_i^{\,2}}{N} - \overline{x}^2 \right) \frac{N}{N-1}$$

  - use (N-1) to compute average in s^2 because we have N independent x_i but if we also know x_{mean} then we only need to know (N-1) x_i to compute last remaining x_i !

    ⇒We call (N-1) the “degrees of freedom” for this calculation

Central Limit Theorem

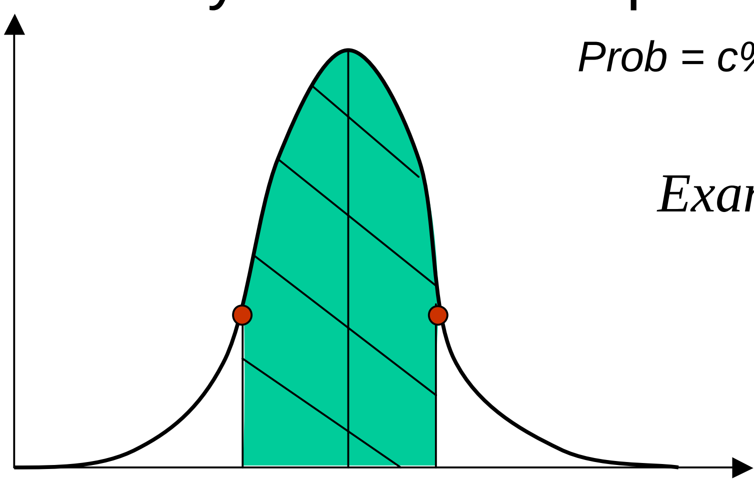
- One of the most important statistical theorems
- Basis for most statistical methods commonly applied to measurements
- **Example**: Suppose we measure a pressure 100 times and the avg. (mean) is 75 psi. Repeat test with 100 more p measurements and get 78 psi. Repeat many times (*N*).
  - **Question:** what is distribution for mean of average values?
  - **Answer**: Gaussian (normal) with where σ is std dev of actual distribution of variable x.

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

- This is also the distribution for any random variable that is the result of many independent random variables - no matter what the underlying distributions may be

Application: Confidence Intervals

- **Question:** If one takes N readings and computes the sample average, how confident can you be that the average is really close to the true mean (μ)?
- **Confidence intervals** are way to describe this; we know that the probability distribution of the sample mean for many different samples (N large) will be normal.



Prob = c% that μ lies in shaded area defined by $\bar{x} = \mu \pm a_c\,\sigma_{\bar{x}}$

$$Example: \quad c = 95\%, \quad a_c = 1.96, \quad \bar{x} = \mu \pm 1.96\,\sigma_{\bar{x}}$$

$$Answer\ 1:\ 95\%\ of\ \bar{x}\ readings\ fall\ in\ \mu \pm 1.96\,\sigma_{\bar{x}}$$

$$Answer\ 2:\ With\ 95\%\ confidence,\ the\ true\ \mu\ will\ fall\ within\ \bar{x} \pm 1.96\,\sigma_{\bar{x}}$$

Application: Confidence Intervals - cont'd (1)

- Since $\sigma_{\bar{x}} = \sigma/\sqrt{N}$ where σ = true population standard deviation, we can write:

$$Answer\ \ 3:\ 95\%\ \ confidence\ \ limits:\ \ \bar{x}\ \pm\ 1.96\ \ \sigma/\sqrt{N}$$

- For large sample sizes (large N), we can approximate σ with s_x so that we have:

$$Answer\ \ 4:\ 95\%\ \ confidence\ \ limits:\ \ \bar{x}\ \pm\ 1.96\ \ s_x/\sqrt{N}$$

<u>EXAMPLE</u>

Find 95% confidence limits for $\bar{x}$ = 75 psi when S_x=8.3 psi for N = 50 samples.

$$Answer:\ limits = \ \bar{x} \pm 1.96\, S_x/\sqrt{N} = 75 \pm 1.96 \frac{8.3}{\sqrt{50}} = 75 \pm 2.3\ psi$$

Find 99.9% confidence limits for previous case (use previous table):

$$Answer:\ limits = \ \bar{x} \pm 3.29\, S_x/\sqrt{N} = 75 \pm 3.29 \frac{8.3}{\sqrt{50}} = 75 \pm 3.9\ psi$$

Application: Confidence Intervals - cont'd (2)

EXAMPLE (cont'd)

How many N are required to assume that mean is in $\bar{x} \pm 5\%$ with 95% confidence?

$$Answer:\ limits = 75 \pm 1.96 \frac{8.3}{\sqrt{N}} = 75 \pm (75 \times 0.05)\,psi$$

$$or:\ N = 18.8 \approx 19\ samples$$

- Remarks
  - Above works only for N=large, or when σ is known
  - When N=small, then we cannot approximate σ by s_x (due to N-1 in the denominator), and must treat σ as unknown
    - This leads to replacing the normal distribution with the "Student T distribution", a subject beyond this introduction

Propagation of Uncertainty

- Many times experimental results are the result of several independent measurements combined using a theoretical formula. For mass flowrate through a pipe: $\dot{m} = \rho u A = \frac{p}{RT} u \pi D^2$
- How do random and bias uncertainties in each variable contribute to whole?

For the case where $y=y(x_1, x_2, \ldots x_N)$ is a linear function, a statistical theorem states that:

$$\sigma_y = \left[\left(\frac{\partial y}{\partial x_1} \sigma_{x_1} \right)^2 + \left(\frac{\partial y}{\partial x_2} \sigma_{x_2} \right)^2 + \cdots \left(\frac{\partial y}{\partial x_N} \sigma_{x_N} \right)^2 \right]^{1/2}$$

For uncertainties, u_i, that are small compared to x_i we can use a Taylor Series expansion in u_i:

$$y(x_1 + u_1, x_2 + u_2, \ldots x_N + u_N) = y(x_1, x_2, \ldots x_N) + \frac{\partial y}{\partial x_1} u_1 + \frac{\partial y}{\partial x_2} u_2 + \ldots \frac{\partial y}{\partial x_N} u_N$$

Now, y is a linear function of the uncertainties and we can use the first equation to yield:

$$u_y = \left[\left(\frac{\partial y}{\partial x_1} u_1 \right)^2 + \left(\frac{\partial y}{\partial x_2} u_2 \right)^2 + \cdots \left(\frac{\partial y}{\partial x_N} u_N \right)^2 \right]^{1/2}$$

Examples of Uncertainty Propagation

- Suppose the output is an additive combination: $y = Ax_1 + Bx_2$

Then $\dfrac{\partial y}{\partial x_1} = A \;\; and \;\; \dfrac{\partial y}{\partial x_2} = B$

and

$$u_y = \left[\left(\frac{\partial y}{\partial x_1} u_{x_1}\right)^2 + \left(\frac{\partial y}{\partial x_2} u_{x_2}\right)^2\right]^{1/2} = \left[A^2 u_{x_1}{}^2 + B^2 u_{x_2}{}^2\right]^{1/2}$$

- Suppose that the output is a multiplicative combination: $y = A\dfrac{x_1{}^m x_2{}^n}{x_3{}^k}$

Then $\dfrac{\partial y}{\partial x_1} = A\dfrac{m x_1{}^{m-1} x_2{}^n}{x_3{}^k} = \dfrac{m}{x_1} A \dfrac{x_1{}^m x_2{}^n}{x_3{}^k} = m\dfrac{y}{x_1}$

$\dfrac{\partial y}{\partial x_2} = n\dfrac{y}{x_2} \;\; and \;\; \dfrac{\partial y}{\partial x_2} = -k\dfrac{y}{x_3}$

and

$$\frac{u_y}{y} = \left[\left(m\frac{u_{x_1}}{x_1}\right)^2 + \left(n\frac{u_{x_2}}{x_2}\right)^2 + \left(-k\frac{u_{x_3}}{x_3}\right)\right]^{1/2}$$

Bias and Precision Uncertainties

- We noted earlier that errors will include both bias (systematic) and precision (random) components
- These can usually be treated as independent and therefore the uncertainties for each can be combined into a total:

$$u_{y-total} = \left[u_{y-precision}{}^2 + u_{y-bias}{}^2 \right]^{1/2}$$

- **Note**: if they are not independent, other combining in other ways may be necessary.
- **SINGLE SAMPLE:** if we are considering a single measurement and calculation of the dependent result (y), we must assume N=1 so that:

$$y = \bar{y} \pm a_c \frac{\sigma}{\sqrt{N}} = \bar{y} \pm a_c \frac{u_y}{\sqrt{1}} = \bar{y} \pm a_c u_y$$

- This affects only the precision error which is purely random so:

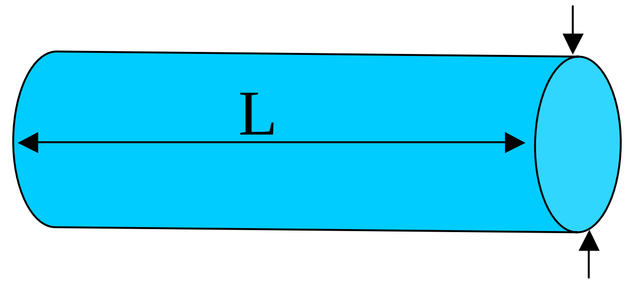
$$u_{y-single\ sample} = a_c u_y = 1.96 u_y$$

$$u_{y-total-single\ sample} = \left[\left(a_c u_{y-precision} \right)^2 + u_{y-bias}{}^2 \right]^{1/2}$$

- Note: above based on 95% confidence interval (1.96 σ), a_c=1.96

Example #1: Uncertainty Calculation

- Consider determining the mass flowrate through a round pipe



$$\dot{m} = \rho \mathrm{v} A = \frac{p}{RT}\mathrm{v}\frac{\pi}{4}D^2 = \frac{p}{RT}\frac{L}{\Delta t}\frac{\pi}{4}D^2$$

- The fractional uncertainty in $\dot{m}$ can then be developed using the previous formula for multiplicative combinations to be:

$$\frac{u_{\dot{m}}}{\dot{m}} = \left[\left(\frac{u_p}{p}\right)^2 + \left(\frac{u_T}{T}\right)^2 + \left(\frac{u_L}{L}\right)^2 + \left(\frac{u_{\Delta t}}{\Delta t}\right)^2 + \left(2\frac{u_D}{D}\right)^2\right]^{1/2}$$

Example #1: Uncertainty Calculation - cont'd

Assume the following uncertainties ($u_{x\text{-precision}} = s_x$)

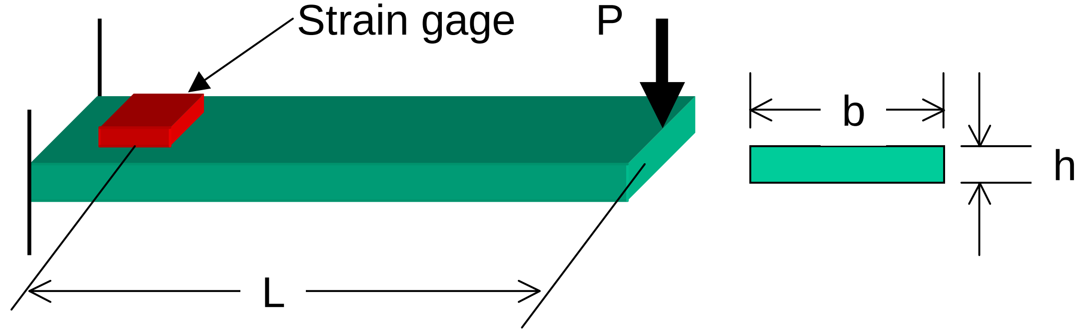
Variable	Bias (u_x/x)	Precision (u_x/x)	Notes
p	0.55%	0.1%	High precision pressure transducer, with 7.5-bit digitizer
T	0.55%	0.4%	Medium precision temp. transducer, with 7.5-bit digitizer
Δt	0.01%	2%	Accurate clock, but starting/stopping uncertainty of 0.01 sec in 0.5 sec measurement
L	0.1%	—	Only measured once with ruler having maximum 0.5 mm reading error over 0.5 m pipe length
D	1%	—	Only measured once with with ruler having maximum 0.5 mm reading error over 50 mm diameter
Summed $(\Sigma u^2)^{1/2}$	1.3%	2.0%	Δt meas. dominates precision error D meas. dominates bias error
Single-Sample	1.3%	4.0%	Precision error dominates

Total uncertainty in single measurement of $\dot{m}$ is then:

$$\frac{u_{\dot{m}}}{\dot{m}} = \left[u_{precision}{}^2 + u_{bias}{}^2 \right]^{1/2} = 4.2\%$$

Example #2: Uncertainty Calculation

- Consider a windtunnel drag transducer made by attaching a strain gage at the root of a cantilever beam as shown. A tip force, P, will produce a bending strain, ε_x, as shown.



$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{M_b \; y}{EI} = \frac{6 \; P \; L}{b \; h^2 \; E}$$

- Assume the electrical output of the strain gage circuit is $e = K\varepsilon_x e_0$ where e_0 is the excitation voltage and K is the calib. factor to be determined.

$$K = \frac{e}{\varepsilon \, e_0} = \frac{e}{e_0} \frac{b \, h^2 \, E}{6 \, P \, L}$$

- The uncertainty in K is then:

$$\frac{u_K}{K} = \left[\left(\frac{u_e}{e} \right)^2 + \left(\frac{u_{e_0}}{e_0} \right)^2 + \left(\frac{u_b}{b} \right)^2 + \left(2 \frac{u_h}{h} \right)^2 + \left(\frac{u_E}{E} \right)^2 + \left(\frac{u_P}{P} \right)^2 + \left(\frac{u_L}{L} \right)^2 \right]^{1/2}$$

Example #2: Uncertainty Calculation - cont'd

Assume the following uncertainties:

Variable	Bias (u_x/x)	Precision (u_x/x)	Notes
e	1%	0.2%	Digital DVM yields good precision
e_0	0.5%	0	Only single initial measurement made
b	0.5%	0	same
h	0.5%	0	same
E	2%	0	same
P	1%	2%	Bias reflects calibration errors while Precision is random error in readings
L	0.2	0	Only single initial measurement made
Uncertainty	2.65%	2.01%	
Single sample	2.65%	3.94%	

Total uncertainty in the measurement of K is then:

$$\frac{u_K}{K} = \left[u_{precision}{}^2 + u_{bias}{}^2 \right]^{1/2} = 4.75\%$$

Wrap-up on Uncertainty Analysis

- Uncertainty analysis should always be a part of the design of any experiment
  - Avoid measurement processes that lead to greater uncertainty
  - Estimate the uncertainties in all measured and computed results
- Consider numerical precision in computational results (since even the best calculator doesn't have infinite precision…)
  - Example: avoid taking differences between very large numbers
  - Example: $s^2 = \sum_{i=1}^{N} \frac{(x_i - \overline{x})^2}{N-1} = \left[\frac{1}{N} \sum_{i=1}^{N} x_i^{\,2} - \overline{x}^2 \right] \frac{N}{N-1}$

Better

Difference between large numbers

- BE CONSISTENT:
  - If the uncertainty in a computed result is 0.1%, then DO NOT include digits beyond 0.1% of the value!
  - Example: if computed result is 13,451.8793 with uncertainty of 0.1%, the number provided in the report should be: 13340.