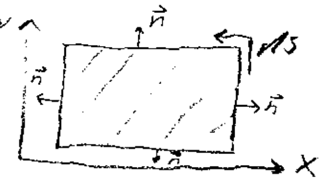
Q: In the Murman & Cole formulation in <u>supersonic</u> flow we approximated

$$\left(\frac{\partial P}{\partial x}\right)_{i,j} \approx \frac{P_{i-\frac{1}{2},j} - P_{i-\frac{3}{2},j}}{\Delta x_i + \underbrace{\Delta x_{i+1}}}$$

↑ Why is Δx_{i+1} used and not Δx_{i-1}?

A: Use of Δx_{i+1} allows the discretized TSD equation to satisfy the divergence theorem (thus conserves mass):



In 2-D we want:

$$\iint (P_x + Q_y)\,dx\,dy = \oint (P\hat{i} + Q\hat{j}) \cdot \vec{n}\,ds$$

- These integrals are evaluated over the entire computational domain to ensure conservation is satisfied globally.
- Applying the right hand side (RHS) just inside the computational domain yields:

$$\oint (P\hat{i} + Q\hat{j}) \cdot \vec{n}\,ds \approx \left[\sum_{j=2}^{jmax} P_{imax-\frac{1}{2},j}\,(\Delta y_j + \Delta y_{j+1})\frac{1}{2} - \sum_{j=2}^{jmax} P_{\frac{1}{2},j}(\Delta y_j + \Delta y_{j+1})\frac{1}{2}\right]$$

$$+\left[\sum_{i=2}^{imax} Q_{i,jmax-\frac{1}{2}}(\Delta x_i + \Delta x_{i+1})\frac{1}{2} - \sum_{i=2}^{imax} Q_{i,\frac{1}{2}}(\Delta x_i + \Delta x_{i+\frac{1}{2}})\frac{1}{2}\right]$$

Consider the 1st term on the left hand side (LHS).

$$\iint P_x \, dx \, dy \approx \sum_{i=2}^{imax} \sum_{j=2}^{jmax} (P_x)_{i,j} \left(\frac{\Delta x_i + \Delta x_{i+1}}{2}\right)\left(\frac{\Delta y_j + \Delta y_{j+1}}{2}\right)$$

$$\approx \frac{1}{4} \sum_{i=2}^{imax} \sum_{j=2}^{jmax} \frac{\overbrace{P_{i-1/2,j} - P_{i-3/2,j}}^{\text{Discretized term for supersonic flow}}}{\frac{1}{2}\cancel{(\Delta x_i + \Delta x_{i+1})}} \cancel{(\Delta x_i + \Delta x_{i+1})}(\Delta y_j + \Delta y_{j+1})$$

Note that these cancel *only* if denominator of $\left(\frac{\partial P}{\partial x}\right)_{i,j}$ is $(\Delta x_i + \Delta x_{i+1})$ not $(\Delta x_i + \Delta x_{i-1})$

* We will see that without this cancellation, the divergence theorem will not *quite* be satisfied.

$$\approx \frac{1}{2} \sum_{j=2}^{jmax} \left[(\Delta y_j + \Delta y_{j+1}) \sum_{i=2}^{imax} \left(P_{i-1/2,j} - P_{i-3/2,j}\right)\right]$$

$$\approx \frac{1}{2} \sum_{j=2}^{jmax} \left[(\Delta y_j + \Delta y_{j+1})\left(\overbrace{\cancel{P_{\frac{3}{2},j}} - P_{\frac{1}{2},j}}^{\text{From } i=2} + \overbrace{P_{\frac{5}{2},j} - \cancel{P_{\frac{3}{2},j}}}^{\text{From } i=3} \cdots\right)\right]$$

Intermediate terms will continue to cancel. (ie. Telescopic Sum)

• Only remaining terms (in this telescopic sum) are the first and last ones:

$$\iint P_x \, dx\, dy \approx \frac{1}{2} \sum_{j=2}^{jmax} (\Delta y_j + \Delta y_{j+1})(P_{imax-1/2,j} - P_{\frac{1}{2},j})$$

$$\approx \left[\sum_{j=2}^{jmax} P_{imax-1/2,j} (\Delta y_j + \Delta y_{j+1})\left(\frac{1}{2}\right) - \sum_{j=2}^{jmax} P_{\frac{1}{2},j} (\Delta y_j + \Delta y_{j+1})\left(\frac{1}{2}\right) \right]$$

○ Note that this exactly equals the 1st bracketed term on page 18.

○ Similarly, we can show that the discretized form of $\iint Q_y \, dx\,dy$ exactly equals the 2nd bracketed term on page 18.

○ Our discretized expression is thus conservative in purely supersonic flow.

○ Can also show that Murman & Cole (1971) is conservative in purely subsonic flow

○ However, a modification was needed to ensure conservation where switching occurs.

• The Murman & Cole (1973) formulation (which is conservative even where A changes signs) is:

$$\boxed{\begin{aligned} &\frac{P_{i+1/2,j} - P_{i-1/2,j}}{\Delta x_i + \Delta x_{i+1}} + \frac{Q_{i,j+1/2} - Q_{i,j-1/2}}{\Delta y_j + \Delta y_{j+1}} \\ &+ \mu_{i-1,j} \frac{P_{i-1/2,j} - P_{i-3/2,j}}{\Delta x_i + \Delta x_{i+1}} - \mu_{i,j} \frac{P_{i+1/2,j} - P_{i-1/2,j}}{\Delta x_i + \Delta x_{i+1}} = 0 \\ &\text{where } \Delta x_i \equiv x_i - x_{i-1} \\ &\qquad \Delta y_j = y_j - y_{j-1} \\ &\qquad \mu_{i,j} = \begin{cases} 1 & \text{if } A_{i,j} \le 0 \quad (\text{supersonic}) \\ 0 & \text{if } A_{i,j} > 0 \quad (\text{subsonic}) \end{cases} \end{aligned}}$$

Murman & Cole (1973 scheme)