Transonic Small Disturbance Equation

- Like in the Full Potential Equation, we assume irrotational, isentropic flow of a perfect gas.
- We also assume that the <u>flow is only slightly perturbed from uniform freestream conditions</u> and seek an equation which is valid near Mach one.
  - The assumption of only slight perturbation means that TSD is limited to <u>slender bodies</u> at only <u>small angles of attack</u> and with only <u>small camber</u>.

- Velocity components can be written:

$$u = \underbrace{V_\infty}_{\text{Freestream}} + \underbrace{u'}_{\text{small disturbance velocity component}} = \phi_x$$

$$v = 0 + v' = \phi_y$$

- Define <u>$\phi' \equiv$ small disturbance velocity potential</u>

$$u' = \frac{\partial \phi'}{\partial x} \qquad v' = \frac{\partial \phi'}{\partial y}$$

- For small disturbances:

$$\frac{u'}{V_\infty} = \frac{\phi_x'}{V_\infty} << 1$$

and $\frac{v'}{V_\infty} = \frac{\phi_y'}{V_\infty} << 1$

- For any size disturbance:

$$\underline{\phi = V_\infty x + \phi'}$$

Note that this satisfies all velocity relations on previous page

- Note also that $\phi_{xx} = \phi'_{xx}$

$\phi_{xy} = \phi'_{xy}$

$\phi_{yy} = \phi'_{yy}$

- We showed previously that the steady, 2-D FPE can be written:

$$\underline{(a^2 - u^2)\phi_{xx} - 2uv\,\phi_{xy} + (a^2 - v^2)\phi_{yy} = 0}$$

- Substitute in 3 previous expressions to get:

$$\underline{(a^2 - u^2)\phi'_{xx} - 2uv\,\phi'_{xy} + (a^2 - v^2)\phi'_{yy} = 0} \qquad (F)$$

* Let's develop a small disturbance approximation for each of the terms in eqn (F)

• First, note that eqn (F) can be written:

$$\left[\left(\frac{a}{V_\infty}\right)^2 - \left(\frac{u}{V_\infty}\right)^2\right]\varphi'_{xx} - 2\left(\frac{u}{V_\infty}\right)\underbrace{\left(\frac{v'}{V_\infty}\right)}_{=\left(\frac{v}{V_\infty}\right)}\varphi'_{xy} + \left[\left(\frac{a}{V_\infty}\right)^2 - \left(\frac{v}{V_\infty}\right)^2\right]\varphi'_{yy} = 0$$

Note that since $\left(\frac{v'}{V_\infty}\right) << 1$ the second term is much smaller than the 1st and 3rd terms and may be neglected.

• Let's now develop a small disturbance approximation for $a^2 - u^2$. From equation (D) we get: $a^2 = a_0^2 - \frac{\gamma-1}{2}(\varphi_x^2 + \varphi_y^2)$

$$a^2 = a_\infty^2 + \frac{\gamma-1}{2}V_\infty^2 - \frac{\gamma-1}{2}(u^2+v^2)$$

$$a^2 - u^2 = a_\infty^2 + \frac{\gamma-1}{2}\left[V_\infty^2 - u^2 - v^2\right] - u^2$$

$$a^2 - u^2 = a_\infty^2 + \frac{\gamma-1}{2}V_\infty^2\left[1 - \left(1+\frac{\varphi'_x}{V_\infty}\right)^2 - \left(\frac{\varphi'_y}{V_\infty}\right)^2\right]$$

$$- V_\infty^2\left[\left(1+\frac{\varphi'_x}{V_\infty}\right)^2\right]$$

Since $\left(\frac{\varphi_x}{V_\infty}\right)^2 \lll 1$ and $\left(\frac{\varphi_x'}{V_\infty}\right)^2 \lll 1$ we set:

$$a^2 - u^2 \approx a_\infty^2 + \frac{\gamma-1}{2} V_\infty^2 \left[1 - 1 - 2\left(\frac{\varphi_x'}{V_\infty}\right)\right]$$
$$- V_\infty^2 \left[1 + 2\left(\frac{\varphi_x'}{V_\infty}\right)\right]$$

$$a^2 - u^2 \approx a_\infty^2 - V_\infty^2 + V_\infty^2 \left[-(\gamma-1)\left(\frac{\varphi_x'}{V_\infty}\right) - 2\left(\frac{\varphi_x'}{V_\infty}\right)\right]$$

$$a^2 - u^2 \approx a_\infty^2 - V_\infty^2 - (\gamma+1) V_\infty^2 \left(\frac{\varphi_x'}{V_\infty}\right)$$

$$\underline{a^2 - u^2 \approx a_\infty^2 \left[\left(1 - M_\infty^2\right) - \underbrace{(\gamma+1) M_\infty^2 \left(\frac{\varphi_x'}{V_\infty}\right)}\right]} \qquad (G)$$

we keep this term since
although it is small
(ie. of order $\frac{\varphi_x'}{V_\infty}$) so is
$(1 - M_\infty^2)$ in transonic flow

- Similarly we can show that

$$\underline{a^2 - v^2 \approx a_\infty^2} \qquad (H)$$

- Now substitute eqns (G) and (H) into eqn (F) and neglecting the 2nd term (as discussed):

$$\boxed{\left[1 - M_\infty^2 - (\gamma+1)M_\infty^2\left(\frac{\varphi_x'}{V_\infty}\right)\right]\varphi_{xx}' + \varphi_{yy}' = 0}$$

Transonic Small Disturbance Eqn (TSD)

- We obtain $\varphi = \varphi(x,y)$ by solving this equation
- Can get u and v from differentiation
- Use isentropic expressions to find local P, T, and ρ from u, v, and known freestream conditions.
    - Use isentropic equations given earlier (in Full Potential Eqn notes) or rederive to get:

$$\boxed{\begin{aligned} P &= P_\infty\left[1 - \frac{\gamma-1}{2}M_\infty^2\left(\frac{V^2}{V_\infty^2} - 1\right)\right]^{\frac{\gamma}{\gamma-1}} \\ \rho &= \rho_\infty\left[\quad '' \quad\right]^{\frac{1}{\gamma-1}} \\ T &= T_\infty\left[\quad '' \quad\right] \end{aligned}}$$

where, in 2-D, $V^2 = u^2 + v^2$