AE 6020 High-Speed Flow HW#2

Solutions

Table 1 Comparison of the results from real and isentropic shock models

Properties	Real shock	Isentropic shock
Mach number, M_2	0.7011	0.6105
Pressure, p_2	0.779	0.9048
Density, ρ_2	1.862	2.115
Velocity, u_2	0.5370	0.4728

(ii) The momentum of the flow in front of the shock is

$$p_1+\rho_1 u_1^2=0.317+1(1)^2=1.317$$

The momentums behind the shock can be calculated for two models.

Real shock : $p_2+\rho_2 u_2^2=0.779+1.862(0.5370)^2=1.316$

Isentropic shock : $p_2+\rho_2 u_2^2=0.9048+2.115(0.4728)^2=1.378$

From the calculation results, it can be concluded that the momentum is conserved for the real shock model but not for the isentropic shock model.

(iii) For calorically perfect gas, the entropy variation can be calculated using thermodynamic relation.

$$s_2-s_1=c_p\ln\frac{T_2}{T_1}-R\ln\frac{P_2}{P_1}=c_p\ln\frac{h_2}{h_1}-R\ln\frac{P_2}{P_1}=R\left(\frac{\gamma}{\gamma-1}\ln\frac{h_2}{h_1}-\ln\frac{P_2}{P_1}\right)$$

The Mach number is related to the enthalpy.

$$h=\frac{1}{\gamma-1}\frac{u^2}{M^2}$$

The enthalpy in front of the shock is

$$h_1=\frac{1}{1.4-1}\times\frac{1^2}{1.5^2}=1.111$$

Real shock : $h_2=\frac{1}{1.4-1}\times\frac{0.5370^2}{0.7011^2}=1.467$

$$s_2-s_1=R\left(\frac{1.4}{1.4-1}\ln\frac{1.467}{1.111}-\ln\frac{0.779}{0.317}\right)=0.0737$$

Isentropic shock : $h_2=\frac{1}{1.4-1}\times\frac{0.4728^2}{0.6105^2}=1.499$

$$s_2-s_1=R\left(\frac{1.4}{1.4-1}\ln\frac{1.499}{1.111}-\ln\frac{0.9048}{0.317}\right)\approx 0$$

Therefore, the entropy is conserved for the isentropic shock model, but not for the real shock model.

iv) No. The potential flow model yields over 10% error in its predictions of P_2, u_2, ρ_2 ...
This is likely unacceptable for most engineering applications. In ~~addition~~ general, when $M_1 \gtrsim 1.2$ the error in FPE is significant.

Exercise 2.3

• From isentropic relations:

$$\rho_0 = \rho\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{1}{\gamma-1}}$$

$$\frac{\rho}{\rho_0} = \left[1+\frac{\gamma-1}{2}M^2\right]^{-\frac{1}{\gamma-1}}$$

$$\frac{\rho}{\rho_0} = \left[1+\frac{\gamma-1}{2}\frac{V^2}{a^{*2}}\frac{a^{*2}}{a^2}\right]^{-\frac{1}{\gamma-1}}$$

$$\frac{\rho}{\rho_0} = \left[1+\frac{\gamma-1}{2}\left(\frac{V}{a^*}\right)^2\frac{\gamma R T^*}{\gamma R T}\right]^{-\frac{1}{\gamma-1}} \quad (A)$$

In adiabatic, steady flow of a perfect gas, $T_0 = \text{constant}$

$$T\left[1+\frac{\gamma-1}{2}M^2\right] = T^*\left[1+\frac{\gamma-1}{2}(1)^2\right]$$

$$\frac{T^*}{T} = \frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}$$

$$\frac{T^*}{T} = \frac{1+\frac{\gamma-1}{2}\left(\frac{V}{a^*}\right)^2\left(\frac{T^*}{T}\right)}{1+\frac{\gamma-1}{2}}$$

$$\frac{T^*}{T}\left[1+\frac{\gamma-1}{2}-\frac{\gamma-1}{2}\left(\frac{V}{a^*}\right)^2\right] = 1$$

$$\frac{T^*}{T} = \frac{1}{1+\frac{\gamma-1}{2}\left[1-\left(\frac{V}{a^*}\right)^2\right]} \quad (B)$$

Substitute eqn (B) into equation (A):

$$\frac{\rho}{\rho_0} = \left\{1+\frac{\frac{\gamma-1}{2}\left(\frac{V}{a^*}\right)^2}{1+\frac{\gamma-1}{2}\left[1-\left(\frac{V}{a^*}\right)^2\right]}\right\}^{-\frac{1}{\gamma-1}}$$

$$\frac{\rho}{\rho_0} = \left\{\frac{1+\frac{\gamma-1}{2}\cancel{-\frac{\gamma-1}{2}\left(\frac{V}{a^*}\right)^2}+\cancel{\frac{\gamma-1}{2}\left(\frac{V}{a^*}\right)^2}}{1+\frac{\gamma-1}{2}\left[1-\left(\frac{V}{a^*}\right)^2\right]}\right\}^{-\frac{1}{\gamma-1}}$$

$$\frac{\rho}{\rho_0} = \left\{\frac{1+\frac{\gamma-1}{2}\left[1-\left(\frac{V}{a^*}\right)^2\right]}{1+\frac{\gamma-1}{2}}\right\}^{\frac{1}{\gamma-1}} = \left\{\frac{1+\frac{\gamma-1}{2}-\frac{\gamma-1}{2}\left(\frac{V}{a^*}\right)^2}{1+\frac{\gamma-1}{2}}\right\}^{\frac{1}{\gamma-1}}$$

$$\frac{\rho}{\rho_0} = \left\{1-\frac{\left(\frac{\gamma-1}{2}\right)(2)(\gamma+1)\left(\frac{V}{a^*}\right)^2}{(2)(\gamma+1)+(\gamma-1)(\gamma+1)}\right\}^{\frac{1}{\gamma-1}} = \left\{1-\frac{(\gamma-1)(\gamma+1)}{2\gamma+2+\gamma^2-1}\left(\frac{V}{a^*}\right)^2\right\}^{\frac{1}{\gamma-1}}$$

$$\frac{\rho}{\rho_0} = \left\{1-\frac{(\gamma-1)\cancel{(\gamma+1)}}{(\gamma+1)\cancel{(\gamma+1)}}\left(\frac{V}{a^*}\right)^2\right\}^{\frac{1}{\gamma-1}} \Rightarrow \boxed{\left(\frac{\rho}{\rho_0}\right) = \left[1-\frac{\gamma-1}{\gamma+1}\left(\frac{V}{a^*}\right)^2\right]^{\frac{1}{\gamma-1}}}$$

(3) Linearized potential eqn for subsonic flow

$$(1-M_\infty^2)\varphi_{xx} + \varphi_{yy} = 0 \qquad (1)$$

let $$\varphi = \frac{\Gamma}{2\pi}\tan^{-1}\left(\sqrt{1-M_\infty^2}\,\frac{y}{x}\right) \qquad (3.28)$$

$$\varphi_x = \frac{\Gamma}{2\pi}\,\frac{-\sqrt{1-M_\infty^2}\cdot\frac{y}{x^2}}{1+(1-M_\infty^2)\frac{y^2}{x^2}}$$

$$= \frac{-\Gamma}{2\pi}\,\frac{y\sqrt{1-M_\infty^2}}{x^2+(1-M_\infty^2)y^2}$$

$$\varphi_{xx} = +\frac{\Gamma}{\cancel{2}\pi}\,\frac{y\sqrt{1-M_\infty^2}}{\{x^2+(1-M_\infty^2)y^2\}^2}\,\cancel{2}x$$

$$= \frac{\Gamma x y\sqrt{1-M_\infty^2}}{\pi\{x^2+(1-M_\infty^2)y^2\}^2} \qquad (2)$$

$$\varphi_y = \frac{\Gamma}{2\pi}\,\frac{\frac{1}{x}\sqrt{1-M_\infty^2}}{1+(1-M_\infty^2)\frac{y^2}{x^2}}$$

$$= \frac{\Gamma}{2\pi}\,\frac{x\sqrt{1-M_\infty^2}}{x^2+(1-M_\infty^2)y^2}$$

$$\varphi_{yy} = \frac{-\Gamma}{\cancel{2}\pi}\,\frac{x\sqrt{1-M_\infty^2}}{\{x^2+(1-M_\infty^2)y^2\}^2}\,\cancel{2}(1-M_\infty^2)y$$

$$= -\frac{\Gamma x y(1-M_\infty^2)^{3/2}}{\pi\{x^2+(1-M_\infty^2)y^2\}^2} \qquad (3)$$

Substitute (2) & (3) into (1) then

$$\frac{\Gamma x y\sqrt{1-M_\infty^2}}{\pi\{x^2+(1-M_\infty^2)y^2\}^2}\left(\cancel{1}-\cancel{M_\infty^2}-\cancel{1}+\cancel{M_\infty^2}\right)=0$$

$\therefore$ Equation (3.28) satisfies the linearized small disturbance equation