III. THE TRANSONIC SMALL DISTURBANCE EQUATION

3.1 Preliminary Remarks

In this chapter, and the next chapter, we consider the techniques for numerically the transonic flow over airfoils and bodies of revolution. Our ultimate interest is in techniques for predicting the pressure distribution Cp and the loads (lift Cl, drag Cd and the pitching moment about the quarter chord cm) over these geometries. For simplicity, we extensively discuss only 2-D planar flows. From the previous chapter, we recall that the governing equations are:

$$\boxed{\left(\rho\phi_x\right)_x + \left(\rho\phi_y\right)_y = 0} \tag{3.1}$$

and

$$\boxed{\frac{\rho}{\rho_\infty} = \left[1 + \frac{\gamma-1}{2}M_\infty^2\left(1 - \frac{u^2+v^2}{V_\infty^2}\right)\right]^{\frac{1}{\gamma-1}}} \tag{3.2}$$

It is possible to eliminate the explicit appearance of density r from equations (3.1) and (3.2), and arrive at the following quasi-linear form of the full potential equation:

$$\boxed{\left(a^2-u^2\right)\phi_{xx} - 2uv\phi_{xy} + \left(a^2-v^2\right)\phi_{yy} = 0} \tag{3.3}$$

Equation 3.3 is calle d quasi-linear because it is linear in its highest derivatives ϕ_{xx}, ϕ_{yy} and ϕ_{xy}. It is, of course, nonlinear in ϕ. It is this nonlinearity that allows us to model shock waves, a very non-linear phenomenon. From a mathematical theory called the method of characteristics, one can show that equation (3.3) is elliptic if

$$(u^2+v^2)/a^2 < 1$$

and hyperbolic if

$$(u^2+v^2)/a^2 > 1$$

and parabolic if

$$(u^2+v^2)/a^2 = 1$$

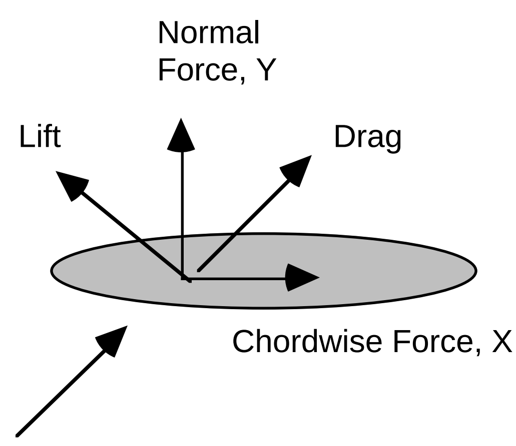
3.2 Loads over the Airfoil

We define the surface pressure coefficient Cp as,

$$\boxed{C_p = \frac{p-p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{\frac{p}{p_\infty}-1}{\frac{\gamma}{2}M_\infty^2} = \frac{\left(\frac{\rho}{\rho_\infty}\right)^\gamma - 1}{\frac{\gamma}{2}M_\infty^2}} \tag{3.4}$$

The airloads over the airfoil may be found once Cp is known as follows:

$$\begin{aligned}
C_Y &= \int_{x=0}^{x=c}\left(C_{p,lower} - C_{p,upper}\right) d\left(\frac{x}{c}\right)\\
C_X &= \int_{x=0}^{x=c}\left[C_{p,upper}\left(\frac{dY}{dx}\right)_{upper} + C_{p,lower}\left(\frac{dY}{dx}\right)_{lower}\right] d\left(\frac{x}{c}\right)\\
C_l &= C_Y\cos\alpha - C_X\sin\alpha\\
C_d &= C_X\cos\alpha + C_Y\sin\alpha\\
C_{m,c/4} &= \int_{x=0}^{x=c}\left[C_{p,upper} - C_{p,lower}\right]\left(\frac{x}{c}-\frac{1}{4}\right) d\left(\frac{x}{c}\right)
\end{aligned} \tag{3.5}$$



where Y(x) is the airfoil shape and α is the angle of attack.

We are interested in the solution of the governing equations, in the entire region between the airfoil and infinity. Because of the nonlinearity of the governing equations, we can not analytically solve the governing equations except in some highly simplified cases. Thus, our approach is necessarily numerical, and is based on CFD techniques.

3.3 Small Disturbance Assumptions

In this chapter, we attempt to simplify the problem by making some physically acceptable assumptions. In the next chapter, we will discard these restrictive assumptions and will solve equation set 3.1 and 3.2 directly.

The assumptions we make are:

(i) The body is thin, has a small angle of attack, and has only a mild camber. As a result, the body slope dY/dx, in a coordinate system attached to the freestream (known as the wind tunnel coordinate system) is small.

(ii) As a consequence, we assume that the local flow velocity components u and v are not significantly different from their freestream values.

It must be noted that the above assumption, and the resulting analysis known as the transonic small disturbance (TSD) theory are, to some extent, only of historical interest. Today, most real world problems are solved by directly solving equations 3.1 and 3.2, known as the full potential equations (FPE). Nevertheless, the TSD theory provides a useful starting point for the more accurate FPE approach.

Disturbance Potential φ:

We introduce a disturbance potential, φ, related to the full potential ϕ as follows:

$$\phi = V_\infty x + \varphi \tag{3.6}$$

It must be remembered that in this chapter we are using the wind tunnel coordinate system, and the freestream velocity is parallel to the x- axis. From equation (3.6), we obtain the flow velocity components terms of φ as follows:

$$\boxed{\begin{aligned} u &= \phi_x = V_\infty + \varphi_x \\ v &= \phi_y = \varphi_y \end{aligned}}$$

(3.7)

From the small disturbance assumptions, it then follows that

$$\boxed{\begin{aligned} \frac{\varphi_x}{V_\infty} \langle\langle 1 \\ \frac{\varphi_y}{V_\infty} \langle\langle 1 \end{aligned}}$$

(3.8)

3.4 Derivation of the Transonic Small Disturbance Equation

Our starting point is the quasi-linear form of the full potential equation 3.3. From equation 3.6 we note that

$$\phi_{xx} = \varphi_{xx}; \phi_{yy} = \varphi_{yy}; \phi_{xy} = \varphi_{xy}$$

(3.9)

Next, consider the second term in equation (3.3), $-2uv\ \phi_{xy}$. This term may be viewed as

$$\begin{aligned} 2uv\varphi_{xy} &= 2\left(V_\infty + \varphi_x\right)\varphi_y\varphi_{xy} \\ &= V_\infty^2\left(1 + \frac{\varphi_x}{V_\infty}\right)\left(\frac{\varphi_y^2}{V_\infty^2}\right)_x \cong 0 \end{aligned}$$

small

(3.10)

Next, consider the coefficient a^2-u^2 , in front of the first term in (3.3). Noting the fact that the speed of sound is related to the flow speed by the energy equation:

$$\boxed{\frac{a^2}{\gamma-1} + \frac{u^2+v^2}{2} = \frac{a_\infty^2}{\gamma-1} + \frac{V_\infty^2}{2}}$$

(3.11)

We can approximate this coefficient as follows:

$$
\begin{aligned}
a^2 - u^2 &= a_\infty^2 + \frac{\gamma-1}{2}\left[V_\infty^2 - u^2 - v^2\right] - u^2 \\
&= a_\infty^2 + \frac{\gamma-1}{2}V_\infty^2\left[1-\left(1+\frac{\varphi_x}{V_\infty}\right)^2 - \left(\frac{\varphi_y}{V_\infty}\right)^2\right] - V_\infty^2\left(1+\frac{\varphi_x}{V_\infty}\right)^2 \\
&\cong a_\infty^2 - V_\infty^2 - (\gamma+1)V_\infty^2\left(\frac{\varphi_x}{V_\infty}\right) \\
&= a_\infty^2\left[1 - M_\infty^2 - (\gamma+1)M_\infty^2\left(\frac{\varphi_x}{V_\infty}\right)\right]
\end{aligned}
$$

(3.12)

Examine the above approximation careful ly. Note that we have neglected second powers of the "disturbance velocities" as small. We have, however, kept the first power of the term φ_x/V_∞ . This is because the term $(1-M_\infty^2)$ itself may be small in transonic flows. Thus, the last term in the above approximation for (a^2-u^2) may be comparable in magnitude to $(1-M_\infty^2)$, and can not be neglected.

In a very similar manner, we can show that

$$
a^2 - v^2 \cong a_\infty^2
$$

(3.13)

With these approximations, the quasi-linear form of the full potential equation takes on the following simpler form:

$$
\boxed{\left[1 - M_\infty^2 - (\gamma+1)M_\infty^2\varphi_x\right]\varphi_{xx} + \varphi_{yy} = 0}
$$

(3.14)

With minor algebraic manipulations, this equation may be written in the following divergence form:

$$
\boxed{\left[\left(1 - M_\infty^2\right)\varphi_x - \frac{\gamma+1}{2}M_\infty^2\varphi_x^2\right]_x + \left[\varphi_y\right]_y = 0}
$$

(3.15)

In computational analyses, it is customary to solve the divergence form (3.15), rather than the quasi-linear form (3.14). The precise reason for this will be discussed later, when we discuss numerical approximations to (3.14) and (3.15).

Equation (3.14) or its equivalent form (3.15) are commonly referred to as the transonic small disturbance equation (TSD).

Exercise 3.1

Show that the axi-symmetric form of the transonic small disturbance equation is given by

$$\left[1 - M_\infty^2 - (\gamma+1) M_\infty^2 \varphi_x\right]\varphi_{xx} + \frac{1}{r}(r\varphi_r)_r = 0$$

3.5 Mathematical Characteristics of the TSD Equation:

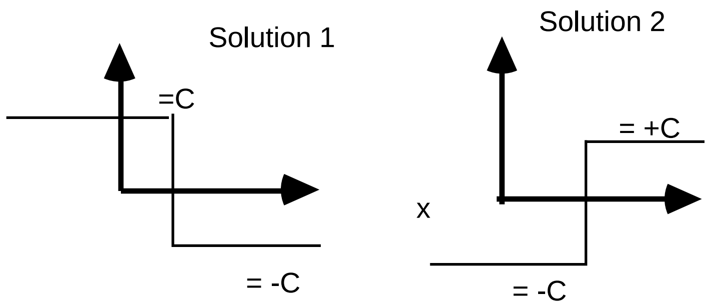
(a) *The transonic small disturbance equation is still nonlinear*, in spite of the approximations that were made to arrive at this equation. It is this nonlinearity which makes formation of shock waves possible. To see why, consider the following equation which is similar in form to equation (3.15):

$$\left(\varphi_x^2\right)_x = 0 \tag{3.16}$$

This equation has the following two solutions:

$$\varphi_x = \pm C \tag{3.17}$$

where C is any constant. These two solutions may occur across a jump (a shock wave) as shown below:



Note that the flow will slow down across the first jump, as the disturbance velocity changes in magnitude from a positive value to a negative value. This type of jump is classified as a compression shock. The second jump, on the other hand, corresponds to an expansion shock, across which the flow abruptly increases in velocity. We therefore conclude that our TSD equation can give rise to compression shocks as well as expansion shocks. *Expansion shocks violate the second law of Thermodynamics, and should be excluded, as part of the numerical solution procedure*.

(b) Equation (3.14) may be elliptic, hyperbolic or parabolic. To see why, write equation (3.14) in the following form:

$$A\varphi_{xx} + \varphi_{yy} = 0 \tag{3.18}$$

where,

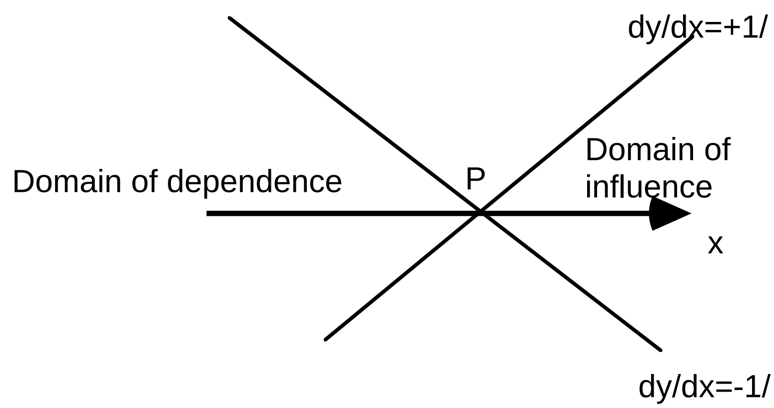
$$A = 1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \varphi_x$$

The character of this equation (i.e. whether it is elliptic, parabolic or hyperbolic) may be determined by finding the roots of the following equation, obtained from the mathematical theory of characteristics:

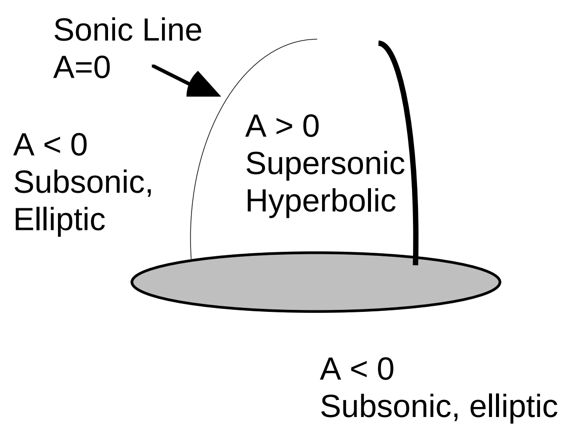
$$\left(\frac{dy}{dx}\right)_{characteristic} = \pm \frac{1}{\sqrt{-A}} \tag{3.19}$$

when A is positive, there are no real characteristics, and the equation is elliptic. When A is zero, the equation is called parabolic, with characteristic lines that are parallel to the y- axis. Finally, when A is negative, two distinct characteristics exist, and the equation is hyperbolic.

At any point P in space, these two characteristics will have slopes that are equal in magnitude and opposite in sign, and will be symmetric about the x-axis, as shown. The region in front of the point P enclosed by the characteristics is known as the domain of dependence of point P. The region downstream of P, enclosed within the characteristics is influenced by point P and known as the domain of influence of P.



In a general transonic flow, the quantity A can change sign from point to point. Thus, the TSD equation may be elliptic in some (subsonic) regions of the flow, parabolic on sonic lines, and supersonic and hyperbolic in other regions:



Any numerical scheme must account for the fact that these three regions may simultaneously exist in a transonic flow. The numerical scheme must be properly designed, so that a point P depends on its entire surrounding when the point is in an elliptic region. The numerical scheme must also ensure that the point P depends only on its domain of dependence in hyperbolic regions. Finally, the numerical scheme must capture the sonic line where the equation is parabolic, and the shock wave, across which the flow is discontinuous.

3.6 Small Disturbance Approximation for Surface pressure Coefficient., C_p:

We next develop a small disturbance approximation for the surface pressure. Our starting point is equations (3.2) and (3.4), which may be combined to yield:

$$C_p = \frac{\left[1+\frac{\gamma-1}{2}M_\infty^2\left(1-\frac{u^2+v^2}{V_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}}-1}{\frac{\gamma}{2}M_\infty^2}$$

$$= \frac{\left[1+\frac{\gamma-1}{2}M_\infty^2\left(1-\frac{\left(V_\infty+\varphi_x\right)^2+\varphi_y^2}{V_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}}-1}{\frac{\gamma}{2}M_\infty^2}$$

$$\approx \frac{\left[1+\frac{\gamma-1}{2}M_\infty^2\left(-2\frac{\varphi_x}{V_\infty}\right)\right]^{\frac{\gamma}{\gamma-1}}-1}{\frac{\gamma}{2}M_\infty^2}$$

In arriving at the above form, we have neglected second powers of terms such as φ_x/V_∞ and φ_y/V_∞ .

Next, we use the binomial expansion

$$(1+\varepsilon)^n \approx 1+n\varepsilon \tag{3.20}$$

where ε is any small quantity $\varepsilon << 1$ and n is any real number. Then,

$$C_p \approx \frac{\left[1-\frac{\gamma}{2}M_\infty^2\varphi_x-1\right]}{\frac{\gamma}{2}M_\infty^2}$$

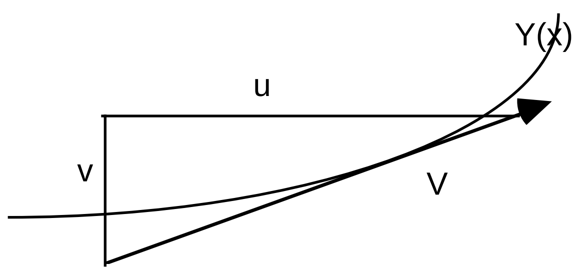
Simplifying, the following small disturbance approximation to surface pressure coefficient results:

$$\boxed{C_p \approx -\frac{2\varphi_x}{V_\infty}} \tag{3.21}$$

3.7 Boundary Conditions

Before the transonic small disturbance equation (3.14) may be solved, we need to specify the boundary conditions. Of course, the boundary conditions must take into account the physics of the problem, and the mathematical characteristics of the equation.

Boundary Conditions at the Solid Boundary:



At nay point on the body surface, the flow must be tangential to the body. In other words, the slope of the velocity vector $\vec{V}$ must equal the body slope.

$$\frac{v}{u} = \frac{\varphi_y}{V_\infty + \varphi_x} = \frac{dY}{dx}$$

Or,

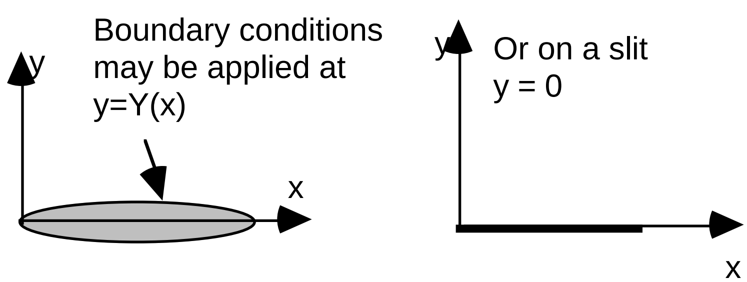
$$\varphi_y = (V_\infty + \varphi_x)\frac{dY}{dx} = V_\infty\left(1 + \frac{\varphi_x}{V_\infty}\right)\frac{dY}{dx}$$

Neglecting the disturbance velocity contribution φ_x/V_∞ in comparison to unity, we get the following boundary condition at the airfoil surface.

$$\boxed{\varphi_y \approx V_\infty \frac{dY}{dx}}$$

(3.22)

Where should this boundary condition be applied? We have two choices. This boundary condition may be applied either at the actual airfoil surface, or on a slit along the chord line, located on the x- axis. The latter choice makes the solution procedure simpler because we can use a Cartesian coordinate system, rather than a curvilinear coordinate system that is wrapped around the body. Within the assumptions built into the small disturbance theory, these two approaches may be shown to be equivalent. Note that this slit is a discontinuity, across which both the disturbance velocity potential φ and its y-derivative are discontinuous.



<u>Boundary Conditions along a cut downstream of the airfoil trailing edge:</u> Consider an airfoil at an angle of attack, producing lift. Then, in a potential flow, the following line integral over any contour enclosing the airfoil will produce a nonzero result, known as the circulation, Γ.

$$\boxed{\oint_{\substack{Around\\Airrfoil}} \vec{V} \bullet d\vec{S} = \Gamma}$$

(3.23)

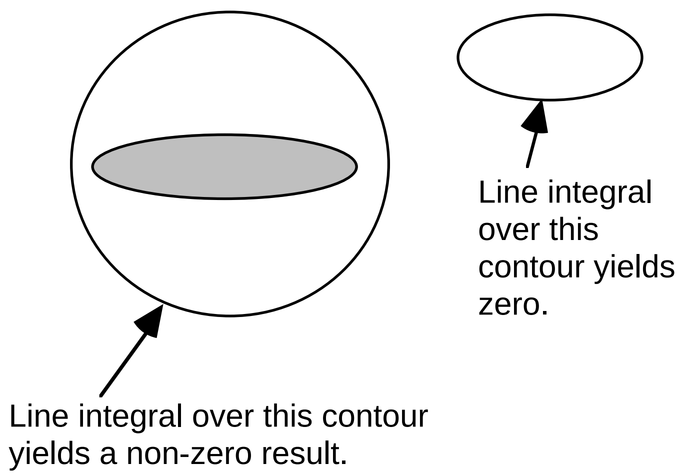
In the above integral, $d\vec{S}$ is an infinitesimal line segment vector, tangential to the contour.

If this contour does not enclose the airfoil, then, using Stokes' theorem the above line integral may be shown to be equivalent to the following area integral:

$$\int_{\substack{\text{Not enclosing}\\ \text{the airfoil}}} \vec{V} \bullet d\vec{S} = \iint_{\substack{\text{Area}\\ \text{enclosed by}\\ \text{the contour}}} \left(\vec{\nabla} \times \vec{V}\right) \bullet \vec{n}\ dA = 0 \tag{3.24}$$

Here $\vec{n}$ is a unit normal to the area element .

The area integral is zero because the curl of the velocity vector is zero in a potential flow.



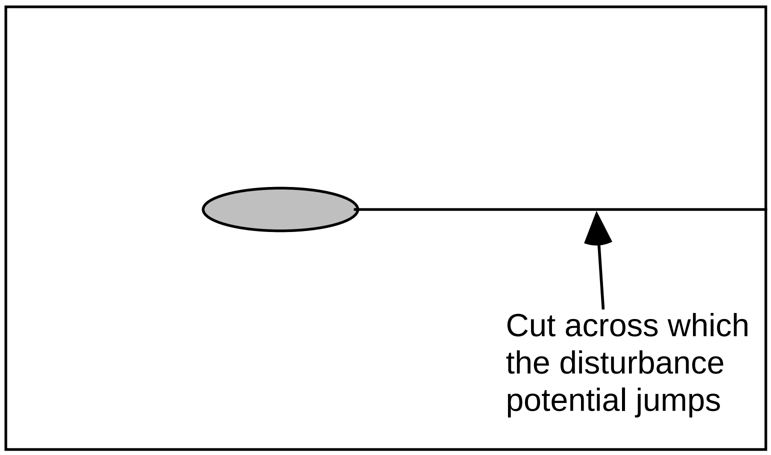
Now, let us link the above integral to the velocity potential. The integrand may be written as

$$\begin{aligned} \vec{V} \bullet d\vec{S} &= \vec{\nabla}\phi \bullet d\vec{S} \\ &= \left(\phi_x \vec{i} + \phi_y \vec{j}\right) \bullet \left(dx\ \vec{i}\ +\ dy\ \vec{j}\right) \\ &= \frac{\partial\phi}{\partial x}\ dx + \frac{\partial\phi}{\partial y} dy \\ &= d\phi \end{aligned} \tag{3.25}$$

Thus, the circulation in equation (3.23) is related to the jump in the velocity potential (or the jump in the disturbance potential):

$$\int_{\substack{\text{Contour}\\ \text{enclosing airfoil}}} \vec{V} \bullet d\vec{S} = \Delta\phi = \Delta\varphi \tag{3.26}$$

The jump in the value of the velocity potential ϕ or the disturbance potential φ implies that these are not single valued functions. Some where in the x-y plane, these functions must experience an abrupt jump in their value, by an amount equal to the circulation, G. The location where the jump occurs may, however, be chosen to suit our convenience. In our analysis, we will assume that the disturbance potential jumps by an unknown amount Γ along a cut that starts at the airfoil trailing edge, and ends at downstream infinity.



Far Field Boundary Conditions:

All numerical calculations are necessarily done on a finite domain. This means the outermost boundary where the disturbance velocity potential is solved for, ends at a finite distance from the airfoil. Boundary conditions consistent with the physics of the flow must be specified on this problem, and over-specification of the boundary conditions must be avoided.

The specification of boundary conditions is different for subsonic freestream ($M_\infty < 1$) and supersonic freestream ($M_\infty > 1$).

Subsonic freestream:

In this case, , the disturbance velocity vanishes at the farfield boundary and the potential flow satisfies the linearized velocity potential equation:

$$\left(1 - M_\infty^2\right)\varphi_{xx} + \varphi_{yy} = 0 \tag{3.27}$$

Since this equation is linear, it may be solved by superposition of sources, sinks, doublets (sources and sinks placed in close proximity with the product of their strength times the separation distance is a constant) and vortices.

For closed bodies, no mass may be generated within the airfoil, or in the flow. Therefore, we should not use source or sink singularities. A doublet (source-sink combination) may be placed somewhere on the airfoil chord line to represent the airfoil thickness effects and the lateral displacement of streamlines. A vortex may also be placed somewhere along the airfoil chord (usually at the quarter chord) to simulate the lift effects.

Now, from incompressible flow, a doublet of strength A produces the following disturbance potential at the far field boundary, at a distance r from the doublet:

$$\varphi \propto \frac{A}{r}$$

Because this quantity rapidly goes to zero at large distances from the airfoil, the doublet effects (i.e. the airfoil thickness effects) are usually not included in the farfield boundaries.

The vortex of strength Γ placed somewhere on the airfoil chord line introduces the following disturbance:

$$\boxed{\varphi = \frac{\Gamma}{2\pi}\arctan\left(\sqrt{1-M_\infty^2}\,\frac{y}{x}\right)} \tag{3.28}$$

This expression may be shown to satisfy equation (3.27). The distances x and y must be measured from the location of the vortex. Note that the above expression reduces to the familiar incompressible form

$$\varphi = \frac{\Gamma}{2\pi}\theta \tag{3.29}$$

associated with point vortices, when the Mach number is zero.

Exercise 3.2 Show that equation (3.28) satisfies the linearized small disturbance equation.

Supersonic Freestream:

When the freestream is supersonic, the uses of sources, sinks and doublets is not appropriate, because signals from the airfoil can not travel upstream. The proper boundary conditions are as follows.

Upstream Boundary: At this boundary, φ is set to zero.

Downstream Boundary: On this boundary, it is not appropriate to specify any boundary conditions, because in supersonic flow information can only flow downstream. In other words, the behavior of φ is determined by the flow over the airfoil itself. Thus, we should determine φ from the nonlinear governing equation or its linearized form, (3.27).

Lateral Boundaries: At these boundaries, the linearized potential equation may be examined to arrive at the proper boundary conditions. From the mathematical theory of characteristics, this equation has two characteristics, and two corresponding compatibility conditions that must be satisfied along the characteristics. These are:

Characteristic 1:

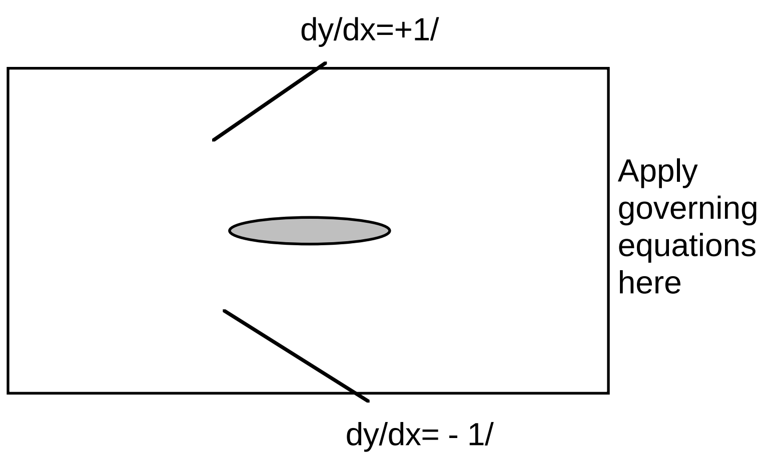
$$\boxed{\begin{aligned} &\frac{dy}{dx} = +\frac{1}{\sqrt{M_\infty^2 - 1}} \\ &\varphi = \text{Constant} \end{aligned}} \tag{3.30}$$

Characteristic 2:

$$\boxed{\begin{aligned} &\frac{dy}{dx} = -\frac{1}{\sqrt{M_\infty^2 - 1}} \\ &\varphi = \text{Constant} \end{aligned}} \tag{3.31}$$

Exercise 3.3: Derive equation set (3.30) and (3.31) from the method of characteristics.

These characteristic equations and their compatibility equations must be applied at the lateral boundaries. Equation set (3.30), which corresponds to a characteristic that starts in the interior of the flow out proceeds upwards and outwards is applied at the top boundary. Equation (3.31) corresponds to a characteristic with a negative slope, which starts at the interior and proceeds down towards the lower lateral boundary. Therefore, this equation is applied at the bottom boundary.



One can apply the method of characteristic at the downstream boundary as well to solve for φ. But it is far easier to solve the governing equations at this boundary, in a manner identical to that used in the interior.

3.8 Concluding Remarks

In this chapter, we derived the equations governing the 2-D transonic small disturbance flow, and the appropriate boundary conditions. The next chapter describes how these equations may be solved using CFD techniques. Chapter IV may be skipped if desired, and the reader may proceed to Chapter V where the full potential equation set is covered.