2. Governing Equations

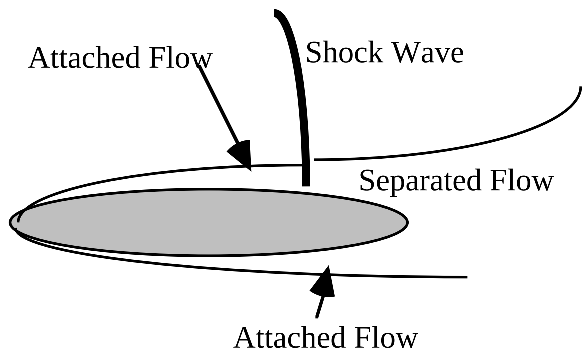


Figure 2.1 Transonic Viscous Flow past an Airfoil

In this chapter, we identify the equation set that is appropriate for analyzing transonic flow. In general, such flows are viscous and unsteady. The boundary layer growth over airfoils, and the possible separation at the foot of the shock waves can adversely affect the airfoil behavior. The shock boundary layer interaction may be unsteady, leading to a phenomenon called "buffeting" where the shock wave oscillates back and forth about a mean position on the airfoil. Such general situations are best described by the unsteady Navier-Stokes equations.

Navier-Stokes Equations:

For the sake of simplicity, we look at 2-D flows first. This flow is governed by the 2-D compressible Navier-Stokes equations. These equations may formally be written as:

$$\frac{\partial q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} \qquad (2.1)$$

where,

q = Flow properties vector, given by $\{ \rho , \rho u , \rho v , E \}$

E = Total (Internal + Kinetic) energy per unit volume

$= \rho C_v T + \rho (u^2 + v^2) / 2$

$F = \{ \rho u , \rho u^2 + p , \rho uv , \rho u h_0 \}$

$G = \{ \rho v, \rho uv , \rho v^2 + p , \rho v h_0 \}$

$R = \{ 0, \tau_{xx} , \tau_{xy} , u\tau_{xx} + v\tau_{xy} + k T_x \}$

$S = \{ o , \tau_{xy} , \tau_{yy}, u\tau_{xy} + v \tau_{yy} + k T_y \}$

h_0 = Specific total enthalpy, $C_p T + (u^2 + v^2) / 2$

k = Conductivity (molecular plus turbulent)

The above equations are parabolic in time. This means, one can solve these equations by marching in time, starting with an initial guess for the flow properties. When q stops changing with time, the steady state solution has been reached. This marching procedure is valid for subsonic, transonic and supersonic flows.

Solving the Navier-Stokes equations is computationally expensive, and requires several minutes of CPU time on Cray Y/MP class of supercomputers, even for simple 2-D problems. Can these equations be further simplified?

Euler Equations

When the shock waves are weak, flow over airfoils (and wings) may be modeled as a viscous-inviscid interaction problem. That is, we can analyze the inviscid flow outside the boundary layer, and the boundary layer itself separately. These two regions then may somehow be linked.

An obvious way to simplify the governing equations in the inviscid regions is to drop the viscous terms that appear on the right hand side. The resulting equations are known as the Euler equations.

$$\frac{\partial q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \tag{2.2}$$

These equations are hyperbolic in time, and may also be solved by marching in time. Solving the Euler equations is easier than solving the Navier-Stokes equations, but still time consuming for routine engineering analyses.

<u>Steady Potential Flow equations:</u>

In search of efficient ways of predicting transonic flows, researchers turned to the potential flow model. In this model, we first assume that the flow is irrotational. If the shock waves are weak, buffeting is not present, and the flow may be assumed steady. A velocity potential Φ can then be defined as follows:

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z} \tag{2.3}$$

Continuity equation then becomes,

$$\frac{\partial\left(\rho \frac{\partial \phi}{\partial x}\right)}{\partial x} + \frac{\partial\left(\rho \frac{\partial \phi}{\partial y}\right)}{\partial y} + \frac{\partial\left(\rho \frac{\partial \phi}{\partial z}\right)}{\partial z} = 0 \tag{2.4}$$

Equation (2.4) has two unknowns, Φ and ρ. Thus all we need is one more equation (preferably algebraic) linking these two unknowns.

Note that the energy equation for steady, inviscid flows may be written as

$$(\rho u h_0)_x + (\rho v h_0)_y + (\rho w h_0)_z = 0$$

When continuity is used, the above equation becomes

$$u\, h_{0x} + v\, h_{0y} + w\, h_{0z} = 0$$

or, in terms of substantial derivative D/Dt,

$$\frac{Dh_0}{Dt} = 0 \tag{2.5}$$

where, the operator D/Dt is defined as

$$D/Dt = \partial/\partial t + u\, \partial/\partial x + v\, \partial/\partial y + w\, \partial/\partial z$$

The physical meaning of equation (2.5) is that the specific total enthalpy h_0 is constant along a streamline. It is assumed that we are only concerned with adiabatic flows (no heat addition), and that there are no body forces doing work on the fluid.

In transonic flows of interest to us, there is no reason to assume that the specific total enthalpy h_0 will vary from one streamline to the next. Thus,

$$h_0 = C_p T + (u^2 + v^2 + w^2)/2 = \text{Constant} \tag{2.6}$$

everywhere in the flow field.

<u>Assumption of isentropy:</u>

In equations (2.4) and (2.6), we have two equations linking three quantities, density ρ , velocity potential Φ and temperature, T. We need a third equation linking these three unknowns to close the system. The isentropic gas law provides a convenient relationship

$$\rho = A\, T^{1/(\gamma-1)} \tag{2.7}$$

where A is a constant.

The isentropic flow assumption, while convenient, has some serious and consequences that limit the potential flow theory to a narrow class of problems. As seen in Figure 2.1, transonic flow usually includes one or more shocks. According to the second law of thermodynamics, across the shock wave, the entropy always rises. Thus, strictly speaking, **transonic flows are never isentropic.**

Fortunately, in the applications that are of interest to aircraft and helicopter industries, strong shock waves are avoided by careful design. The Mach numbers ahead of the shock wave seldom exceeds 1.2 in these flows. The shock wave is very weak, and the entropy rise across weak shock waves is negligible. Thus, isentropic flow approximations are acceptable when weak shock waves alone are present.

Exercise 2.1

Show that in external flows the energy equation and the isentropic gas relations may be combined to yield

$$\frac{\rho}{\rho_\infty} = \left[1 + \frac{\gamma - 1}{2} M_\infty^2 \left(1 - \frac{u^2 + v^2 + w^2}{V_\infty^2}\right)\right]^{\frac{1}{\gamma - 1}}$$

Shock waves of varying strength also produce vorticity behind the shock wave. According to Crocco's theorem, when there is an entropy gradient, there is also vorticity. Again, for weak shock waves, (i.e. if the local Mach number ahead of shock waves is less than 1.2), this vorticity production may be neglected, and the flow may be assumed irrotational.

In summary, the isentropic potential flow approximation is applicable to transonic flows with weak shock waves. This theory reduces to the following set of equations:

$$\begin{aligned}
&\frac{\partial(\rho\phi_x)}{\partial x} + \frac{\partial(\rho\phi_y)}{\partial y} + \frac{\partial(\rho\phi_z)}{\partial z} = 0 \\
&h_0 = C_p T + \frac{u^2 + v^2 + w^2}{2} = \text{const} \\
&\rho \propto T^{\frac{1}{\gamma - 1}}
\end{aligned} \qquad (2.8)$$

If the shock waves are not weak, the isentropic potential flow model given by equation set (8) will cheerfully give results, but the results will be garbage. Therefore, whenever isentropic potential flow approximations are used, it is the user's responsibility to check that the local Mach numbers ahead of shock waves never exceed 1.2. If this occurs, then the Euler equation set must be used.

Conservation of Momentum:

In deriving the isentropic potential flow model derived above, we made use of the continuity equation and the energy equation. We did not use the u-, v- or w- momentum equations. The question then arises: *Does the isentropic flow model satisfy the conservation of momentum?*

In order to answer the question, consider the continuity equation:

$$(\rho u)_x + (\rho v)_y + (\rho w)_z = 0$$

Multiply this equation by u and integrate by parts:

$$(\rho u^2)_x + (\rho uv)_y + (\rho uw)_z - \rho u u_x - \rho v u_y - \rho w u_z = 0 \qquad (2.9)$$

Use the condition of irrotationality :

$$u_y = \partial(\Phi_x)/\partial y = \partial(\Phi_y)/\partial x = v_x$$
$$u_z = \partial(\Phi_x)/\partial z = \partial(\Phi_z)/\partial x = w_x$$

Equation (2.9) becomes.

$$(\rho u^2)_x + (\rho uv)_y + (\rho uw)_z - \rho/2 \ [(u^2 + v^2 + w^2)_x] = 0$$

Use energy equation (2.6) into the above equation:

$$(\rho u^2)_x + (\rho uv)_y + (\rho uw)_z + \rho C_p T_x = 0 \qquad (2.10)$$

When the isentropic gas law is differentiated with respect to x, we get,

from the isentropic Gas law linking p and T:

$$p = B\,T^{\gamma/(\gamma-1)}$$

$$\partial p/\partial x = [\,p / T\,]\ [\,\gamma/(\gamma-1)\,]\ T_x = \rho C_p\, T_x$$

Using the above relationship in equation (10) we recover the u- momentum equation:

$$\frac{\partial\left(\rho u^2 + p\right)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = 0 \qquad (2.11)$$

By a similar process, by multiplying the continuity equation by 'v' and 'w', we can recover the v- and w- momentum equations respectively.

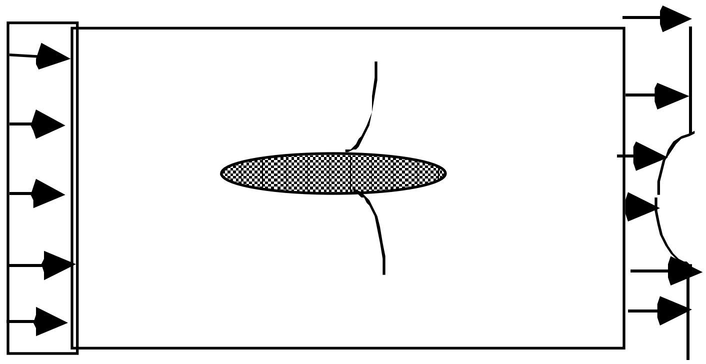
Thus, in smooth regions of the flow, the isentropic potential flow model automatically and exactly satisfies the conservation of momentum.

Why did we use the qualifier, "in smooth regions of the flow" ? This is because, when shock waves are present, across shock waves, terms such as $\partial u/\partial x$, $\partial v/\partial x$, $\partial p/\partial x$ etc. all "blow up" because velocity, density, temperature and pressures are discontinuous across shock waves. Thus, the above mathematical manipulations are not valid in the vicinity of, and across shock waves. We therefore conclude

Across shock waves, the isentropic potential flow model does not conserve momentum.

Origin of Wave Drag:

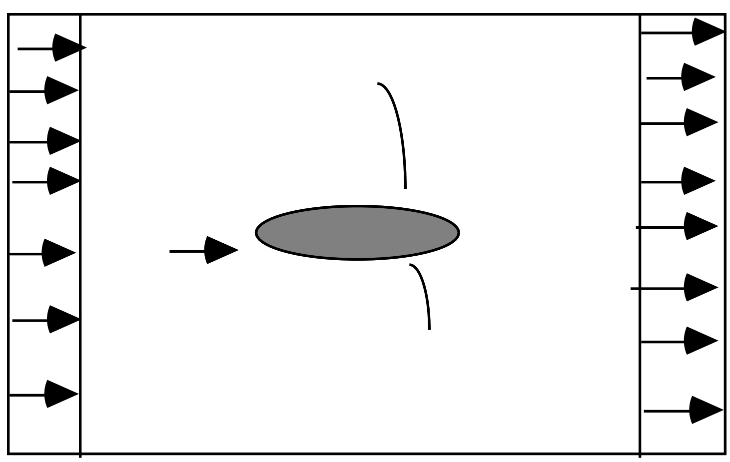
In transonic flow past airfoils, wings, rotors, propellers and so on, a quantity of great interest is the wave drag. We ask the following questions: (a) What is wave drag? (b) How do the Euler equations and the isentropic potential flow equations model wave drag?



Behind the shock there is a total pressure loss. Thus velocity never to its original freestream conditions.

Figure 2.3 Origin of Wave Drag in Euler Equations

The Euler equations and the potential flow model give rise to wave drag for entirely different reasons. As shown in Figure 2.3, Euler equations correctly predict a loss in total pressure across shock waves. As a result, at the downstream boundary, the pressure may recover to the free stream condition, but the velocity can never recover to the free stream conditions. This gives rise to a wake-like velocity profile at the downstream boundary, even when viscous effects are ignored. **It is this velocity deficit, caused by total pressure drop, which gives rise to wave drag.**



Momentum in equals Momentum out at far field boundaries, but not across shock waves.

Figure 2.4 Origin of Wave Drag in Potential Flow is associated with the Change in u- Momentum across the Shock Wave

Potential flow equations do not allow total pressure losses across shock waves, because entropy (and hence total pressure) is conserved. As a result, as shown in Figure 2.4, at the downstream boundary both the velocity and pressure recover back to the free stream values. If we modify our control volume boundaries to include the airfoil and the shock waves, however, we find that the pressure forces over the airfoil equal the integrated values of momentum fluxes and pressure forces over the shock waves. As stated earlier, across shock waves, potential flow model does not conserve momentum. Thus, the **Potential flow model spuriously gives rise to a wave drag, as a result of its inability to conserve momentum across shock waves.**

In summary, Euler equations correctly model wave drag. Potential flow equations spuriously model wave drag.

Baldwin and Steger at NASA Ames Research Center have further studied the wave drag produced by shock waves in potential flow. They have demonstrated that *for weak shock waves, the wave drag predicted by the potential flow model is acceptably close to that from an Euler analysis*. If the shock waves are strong, these two models differ, and the potential flow model fails to give the correct wave drag.

Isentropic Shocks vs. "Real" Shocks

How does the isentropic flow model a shock wave? And, how does this shock wave differ from the "real" shock wave modeled by the Rankine-Hugoniot relations? To answer this question, consider the following 1-D shock wave. The flow ahead of the shockwave is uniform. Let us normalize the flow by conditions ahead of the shock wave.

$\rho_1=1$	$\rho_2=?$
$u_1=1$	$u_2=?$
$M_1=1.2$	$M_2=?$
$p_1=\rho_1 R T_1=1/(\gamma M_1^2)$	$p_2=?$

The "real" shock wave will satisfy the following jump conditions:

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$
$$h_{0_1} = h_{0_2}$$

We can find from the normal shock tables, or from the above equations, condtions behind the shock:

$$\rho_2 = 1.3416$$
$$p_2 = 0.7506$$
$$u_2 = 0.7454$$

Note that

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 = 1.4960$$

Thus, momentum is conserved across the shock wave. One can similarly show that the stagnation enthalpy h_0 is total across the shock wave.

The quantity p/ρ^γ is a measure of the entropy. In front of the shock wave, this quantity is 0.4960. Behind the shock wave, this quantity is 0.4974. Thus entropy rises across a shock wave, as expected. For such a weak shock, the rise in entropy is small.

Now, let us see how the "potential flow" model will model such as hock wave. The equations to be solved are:

<u>Continuity:</u>

$$\rho_1 u_1 = \rho_2 u_2$$

<u>Conservation of Energy:</u>

$$C_p T_1 + \frac{u_1^2}{2} = C_p T_2 + \frac{u_2^2}{2}$$

<u>Conservation of Entropy:</u>

$$\frac{T_1}{T_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma-1}$$

Since we already know ρ_1, u_1 and since

$$C_pT_1 = a_1^2/(\gamma-1) = 1/[M_1^2(\gamma-1)]$$

we can solve for ρ_2, u_2 and T_2 by the following iterative process:

(a) Guess a value for ρ_2

(b) From the continuity equation find u_2.

(c) From energy equation, find C_pT_2

(d) From conservation of entropy, find ρ_2. Now go back to step (a).

When steps (a) through (d) are repeated, the iteration quickly converges to the following values:

$$u_2 = 0.729, \quad \rho_2 = 1.372, \quad M_2 = 0.822$$

Note that:

(i) The Mach number behind the shock is slightly lower than from the normal shock relations, the pressure and density behind the shock wave are slightly higher than from the normal shock tables. Thus, the isentropic shock is stronger than the real shock wave.

(ii) Across the isentropic shock, p/ρ^γ is constant as expected.

(iii) The quantity $p+ \rho u^2$ is not conserved across the shock wave. In front of the shock it is 1.496, while behind the shock wave it is 1.500.

(iv) From Mach numbers close to unity (1.2 in our case), the isentropic shock is reasonably close to the real shock, in so far as density, pressure and velocity are concerned.

Thus, we conclude that the isentropic potential flow is a fairly good model of real world flow, modeled by Euler equations.

Exercise 2.2

Repeat the above analysis for the case where $M_1 = 1.5$

(i) What are the conditions behind the shock from the real and isentropic shock models?
(ii) Is momentum conserved across the shock in these two models?
(iii) Is entropy conserved across these two models?
(iv) Will you use a potential flow model to model <u>this</u> shock wave?

Exercise 2.3

In some potential flow calculations, the density is normalized with respect to the stagnation density ρ_0, and the velocity is normalized with respect to the critical speed of sound a_*. In such a situation, show that the density is given by

$$\rho = \left[1 - \frac{\gamma - 1}{\gamma + 1}\left\{u^2 + v^2 + w^2\right\}\right]^{\frac{1}{\gamma - 1}}$$